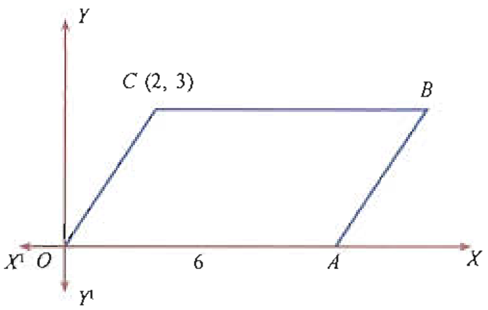
# 10 Geometry and Algebra

## Distance

We saw that by choosing a pair of perpendicular lines and a unit to measure length, we can represent all points in a plane as pairs of numbers.

In the figure below, $OABC$ is a parallelogram.

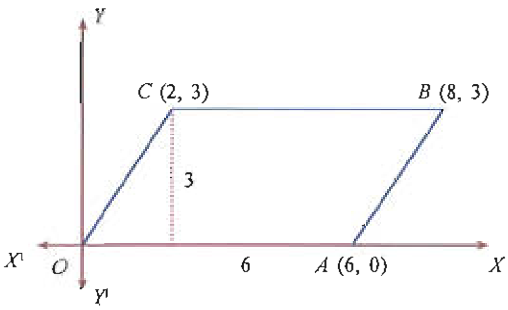



Can you find the coordinates of the vertices $A$ and $B$?

The vertex $A$ is on the $x$-axis itself and its distance from the origin is 6. So what are its coordinates?

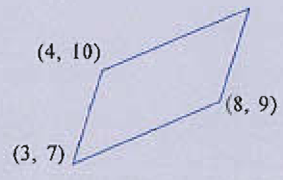
What about $B$? The line $BC$ is parallel to the $x$-axis; and the point $C$ on it has $y$-coordinate 3. So, what's the $y$-coordinate of $B$?

Now the length of $BC$ is also 6. (How do we get that?) So, what's the $x$-coordinate of $B$?

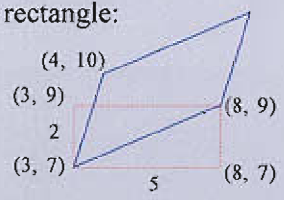



### Fourth vertex

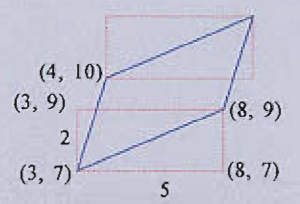
What are the coordinates of the fourth vertex of the parallelogram shown below?



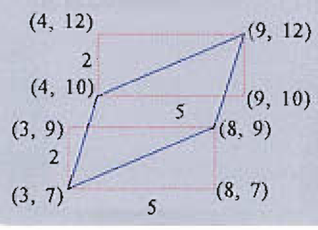

We can draw lines parallel to the axes through the bottom vertices and make a rectangle:




Similarly we can draw a rectangle through the top vertices also:

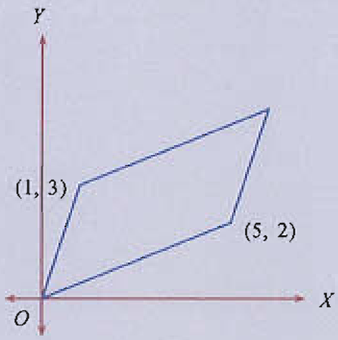



Its width and height are the same as those of the bottom rectangle (why?)

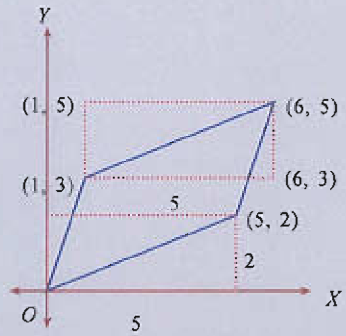


## Parallelogram addition

What are the coordinates of the fourth vertex of this parallelogram?




Let's draw rectangles as before:

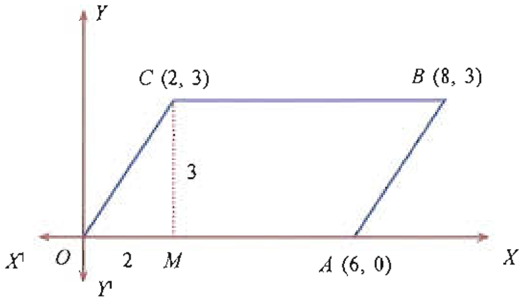



Try this with other points instead of (1, 3) and (5, 2). Do you see any relation between the coordinates we start with and the coordinates of the fourth vertex?

Try with the starting points as $(x_1, y_1)$ and $(x_2, y_2)$.

Now another question: what's the length of the other side of this parallelogram?

See this picture:




How do we get $OM = 2$ ?

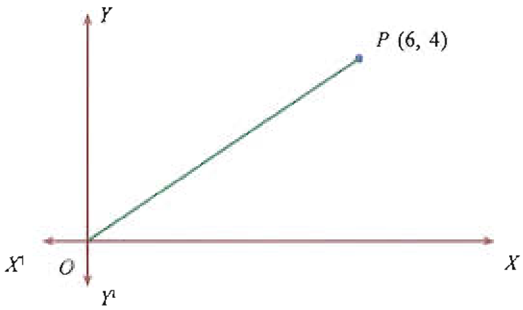
Now from the right angled triangle $COM$, can't we find $OC$?

$$OC^2 = OM^2 + MC^2 = 13$$

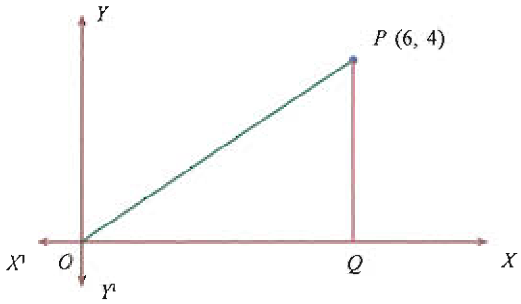
This gives the length of the other side of the parallelogram as $\sqrt{13}$ .

Here we used only the coordinates of $C$ to compute the length $OC$, right?

Like this, can you find the length $OP$ in the figure below?




Draw the perpendicular from $P$ to the $x$-axis.

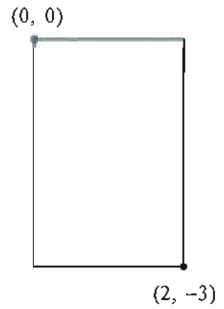



What are the lengths of the perpendicular sides of the right angled triangle $OPQ$? So, can't you find $OP$?

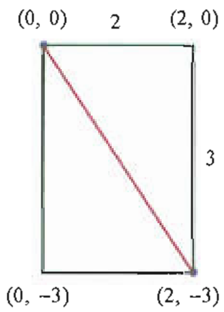
Now suppose the coordinates of a point are given in a figure in which the axes are not shown. Can you find the distance of this point from the origin?

For example, what is the distance between the origin and the point (2, −3)?

For this, we draw a rectangle with its sides parallel to the axes, as shown below:




What are its other vertices? And the lengths of its sides? So, can't we find the length of the diagonal?

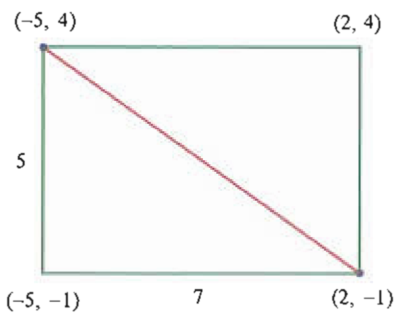



Thus the distance we seek is

$$\sqrt{4+9} = \sqrt{13}$$

Can we find the distance between any two points like this?

For example, let's take (2, −1) and (−5, 4). In this case, we can draw a rectangle like this, with sides parallel to the axes:




### Algebra of geometry

We usually state general relations between numbers using algebra. And we have also seen how some such relations between positive numbers can be geometrically described.

Once we have represented all points in a plane as pairs of numbers, we can describe the geometric relations between these points and the geometric figures formed by joining these points, in the language of algebra.

One such example we have already seen: if $(x_1, y_1)$ and $(x_2, y_2)$ are joined to (0, 0) and the figure is completed to a parallelogram, then the fourth vertex is $(x_1 + x_2, y_1 + y_2)$.

The distance we need is the length of the diagonal of this rectangle; which we can find as

$$\sqrt{7^2+5^2}=\sqrt{74}$$

Now let's take points $(x_1, y_1)$ and $(x_2, y_2)$ with $x_1 \neq x_2$ and $y_1 \neq y_2$ instead of specific points like (2, –1), (–5, 4). Then also, we can draw a rectangle with these as opposite vertices and sides parallel to the axes. A pair of adjacent sides of this rectangle are the lines joining $(x_1, y_1)$, $(x_2, y_1)$ and $(x_1, y_1)$, $(x_1, y_2)$. The lengths of these lines are $|x_1-x_2|$ and $|y_1-y_2|$. So, the square of the length of its diagonal is

$$|x_1-x_2|^2+|y_1-y_2|^2=(x_1-x_2)^2+(y_1-y_2)^2$$

(Do you remember seeing in the lesson **Real Numbers** of the Class 9 textbook that the square of a number is equal to the square of its absolute value?)

Thus the distance between the points $(x_1, y_1)$ and $(x_2, y_2)$ is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

Now a rectangle like this is possible, only if the line joining the points chosen is not parallel to either axis. But we have already seen how we can compute the distance between such points. (The lesson **Coordinates**)

Let's write those also in algebraic terms: .

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- If the $y$-coordinates of two points are equal, as in $(x_1, y)$ and $(x_2, y)$, then the line joining them is parallel to the $x$-axis; and the distance between them is got by subtracting the smaller of $x_1, x_2$ from the larger, that is $|x_1-x_2|$.
- If the $x$-coordinates of two points are equal, as in $(x, y_1)$ and $(x, y_2)$, then the line joining them is parallel to the $y$-axis; and the distance between them is got by subtracting the smaller of $y_1, y_2$ from the larger, that is $|y_1-y_2|$.

Thus we have three formulas to compute the distance between two points, depending on their position.

Now what do we get if we take $y_1 = y_2$ in the algebraic expression $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ ?

### Force parallelogram

We can produce the same effect of two forces acting along different directions on a body, by a single force acting along a definite direction.

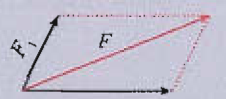
There is a method, recognized through experiments, to find this force and its direction. Draw two lines from a point with their lengths proportional to the forces (such as for example one centimetre for one Newton), along the directions of the forces:

$F_2$

Next draw a parallelogram with these as adjacent sides:

$F_2$

The single force to replace these two forces acts along the diagonal of this parallelogram; and its magnitude is the length of this diagonal, in the scale chosen.




$F_2$

This is known as the Parallelogram Law of Forces.

$$\sqrt{(x_1 - x_2)^2} = |x_1 - x_2|$$

(See the section **Square root and absolute value** of the lesson **Real Numbers**, in the Class 9 textbook.)

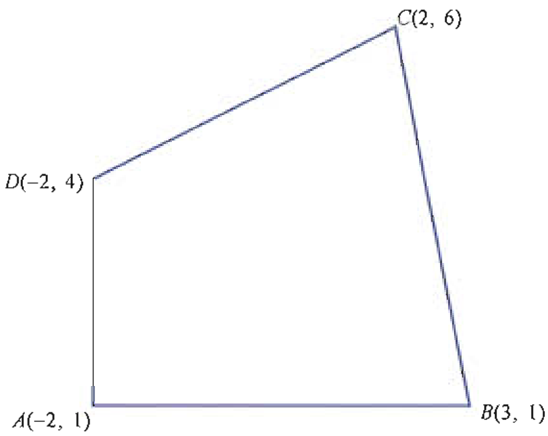
Likewise, if we take $x_1 = x_2$ in the expression, we get $|y_1 - y_2|$.

Thus the distance between two points can be given by a single algebraic expression.

*The distance between any two points $(x_1, y_1)$ and $(x_2, y_2)$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$*

Let's look at some examples: .

- Find the perimeter of the quadrilateral shown below:




Here, the points $A$ and $B$ have the same $y$-coordinates. So, the length of $AB$ is $3 - (-2) = 5$

The points $A$ and $D$ have the same $x$-coordinate and so the length of $AD$ is $4 - 1 = 3$.

The points $B$ and $C$ have different $x$-coordinates and different $y$-coordinates. So, the length of $BC$ is $\sqrt{(2-3)^2 + (6-1)^2} = \sqrt{26}$

Similarly, the length of $CD$ is $\sqrt{(2-(-2))^2 + (6-4)^2} = \sqrt{20}$

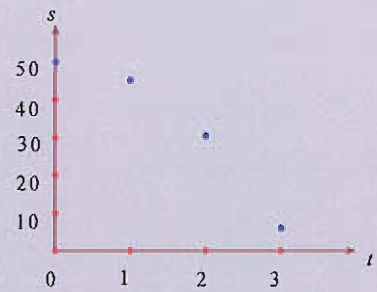
Now we can find the perimeter as $5 + 3 + \sqrt{26} + \sqrt{20} = 8 + \sqrt{26} + 2\sqrt{5}$

### Relations in physics

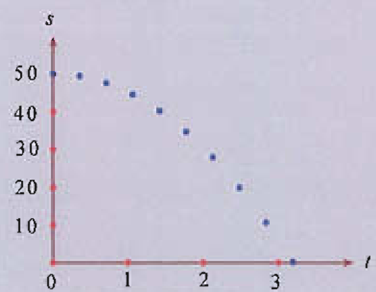
We have seen how algebra is used to describe relations between various physical quantities. For example, if an object falling towards the ground from a height of 50 metres is at a height $s$ metres after $t$ seconds, then

$$s = 50 - 4.9t^2$$

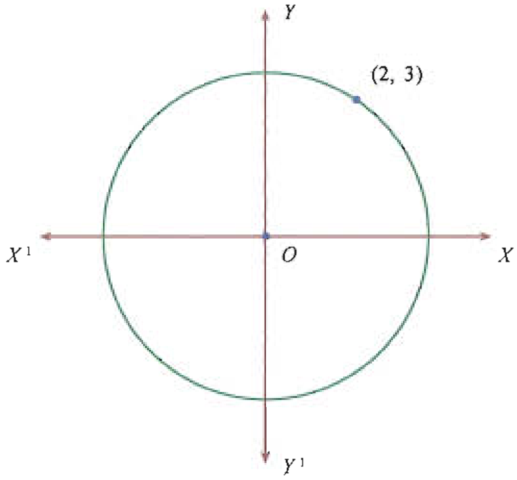
If we take $t = 0, 1, 2, 3$, in this equation, we get $s = 50, 45.1, 30.4, 5.9$: If we draw two perpendicular lines and mark $s$ and $t$ along them using suitable scales, then we can plot the points (0, 50), (1, 45.1), (2, 30.4), (3, 5.9). And we get a picture like this:




Taking more numbers as $t$ and plotting more points, we get a picture like this:



- The circle shown below is centred at the origin. What is its radius?




$O$ is the centre of the circle and (2, 3) is a point on the circle. So, the radius is the distance between them.

What are the coordinates of $O$?

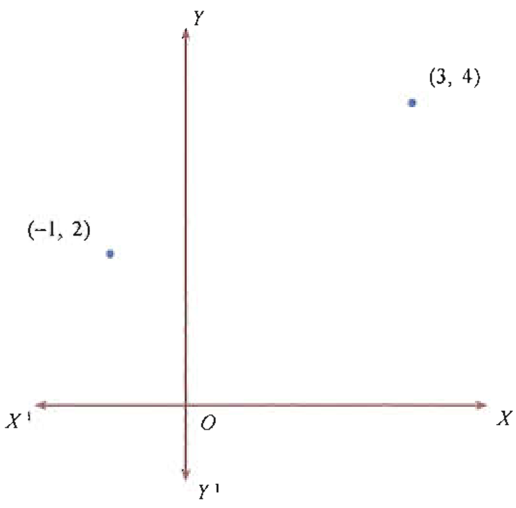
So, the radius is

$$\sqrt{(2-0)^2+(3-0)^2}=\sqrt{13}$$

We can note another thing here. The distance between (3, 2) and the origin is also $\sqrt{13}$. (How is that?) So, this point is also on the circle.

Can you quickly find a few more points on the circle? Mark them on the circle.

- In the figure below, find the point on the $x$-axis, equidistant from the two points marked:




## Electronic aid

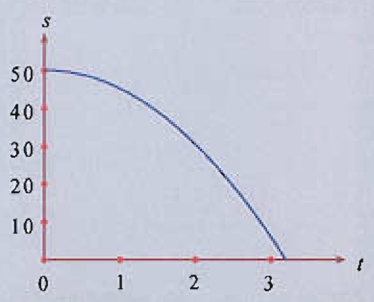
We saw how the relation between the height and time of a falling body is geometrically pictured. In drawing such pictures, it is not easy to compute a large number of coordinates and to plot them.

There are numerous computer applications which can be used for this, such as GeoGebra, Gnuplot, Kmplot. We need only specify the equation of the relation for these to plot the picture.

Given below is the plot of the equation

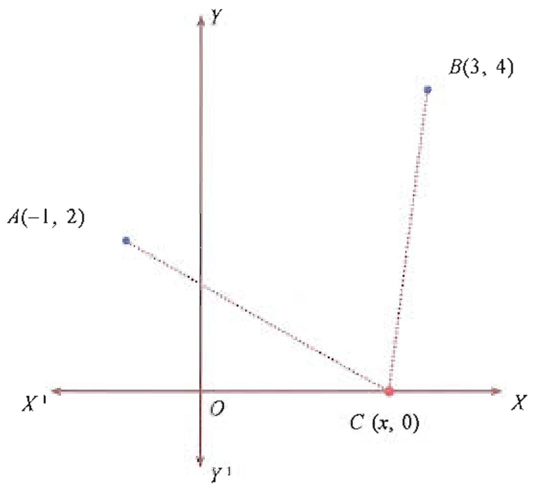
$$s = 50 - 4.9t^2$$

seen earlier, drawn using the PostScript language:




From this picture, we can see such things as how the height decreases with time and when the body would hit the ground.

The required point is on the $x$-axis and so its $y$ - coordinate is zero. If we take its $x$-coordinate as $x$, then its coordinates are $(x, 0)$.




Now in the figure, we have $AC = BC$ which means $AC^2 = BC^2$. Thus

$$(x + 1)^2 + (0 - 2)^2 = (x - 3)^2 + (0 - 4)^2$$

From this, we get

$$(x + 1)^2 - (x - 3)^2 = 12$$

Simplifying, we get

$$8x - 8 = 12$$

and this gives $x = \frac{5}{2}$. So, the coordinates of the point we seek are $(\frac{5}{2}, 0)$.

Now some problems for you:

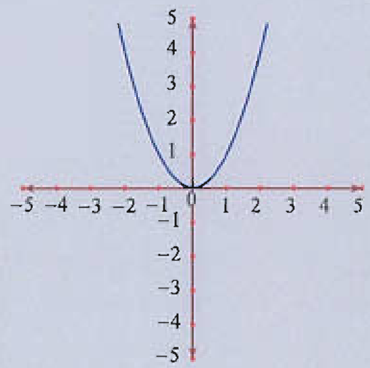
- The centre of a circle is (3, 4) and it passes through the point (2, 5). What is its radius?
- A circle of radius 3 is drawn with centre at (−2, 1). Find out whether the point (4, 1) lies on the circle, within the circle or outside the circle.
- Prove that we get a right angled triangle by joining the points (2, 1), (3, 4), (−3, 6).
- How many points are there on the $x$-axis, at a distance 5 from the point (1, 3)? What are their coordinates? What about such points on the $y$-axis?

## Pictures of equations

Using computers, we can also plot equations giving relations between numbers, For example, the picture below is the plot of the equation

$$y = x^2$$

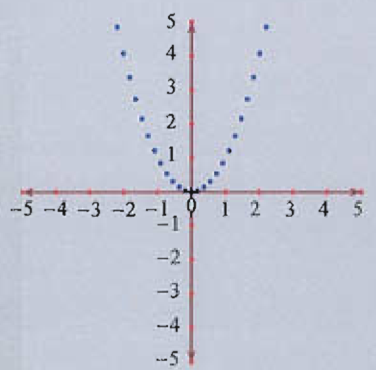
done using PostScript:



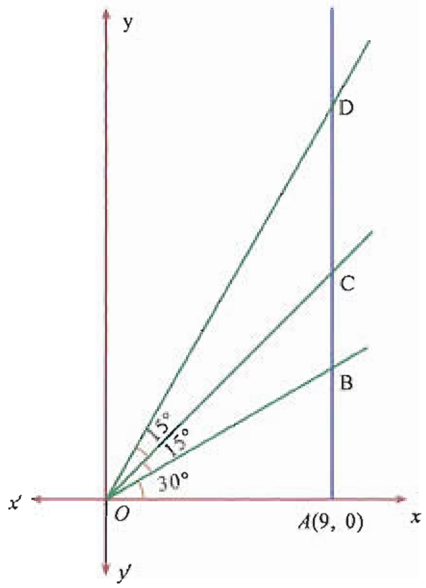

What is the meaning of this picture?

By taking different numbers as $x$, we can calculate the corresponding numbers $x^2$. The plot is made by joining a large number of such pairs $(x, x^2)$, such as for example (1.5, 2.25).

The picture above is made up of 50 such points. If only 25 points are plotted, the plot would look like this:



- Find the coordinates of the points $B$, $C$, $D$ in the picture below:




Write the lengths $AB$, $BC$, $CD$ in the order of their magnitudes.

- The circle centred at (2, 3) and of radius 5, intersects the $x$-axis at $A$ and $B$. Find the coordinates of $A$ and $B$ and also the length of the chord $AB$.

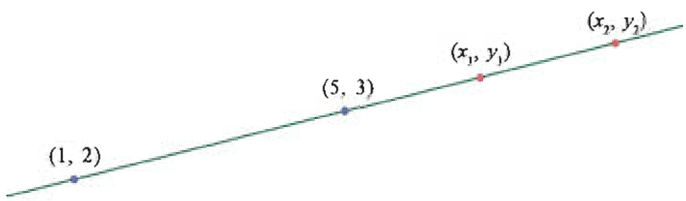
- The vertices of a triangle are the points (1, 2), (2, 3), (3, 1). Find the centre and radius of its circumcircle.

## Slope of a line

We have seen the peculiarities of the coordinates of points on lines parallel to the axes of coordinates: the $y$-coordinates of points on a line parallel to the $x$-axis are equal; and the $x$-coordinates of points on a line parallel to the $y$-axis are equal.

What about points on lines not parallel to either axis? Do the coordinates of points on them have any speciality?

For example, let's look at the line joining the points (1, 2), (5, 3) and two other points on this line:

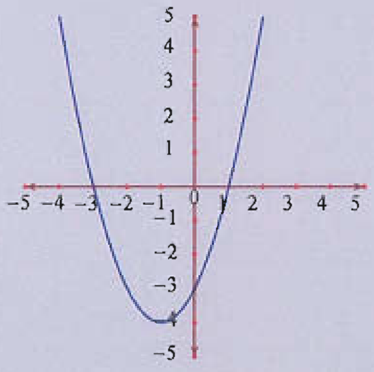



### Second degree plot

The plot below is that of the equation

$$y = x^2 + 2x - 3$$

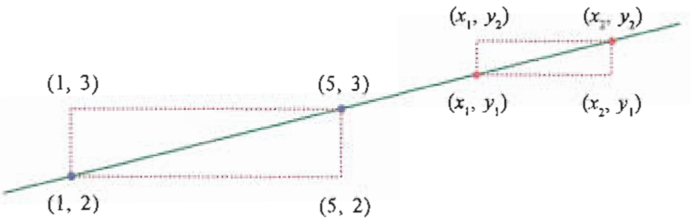
done using a computer:



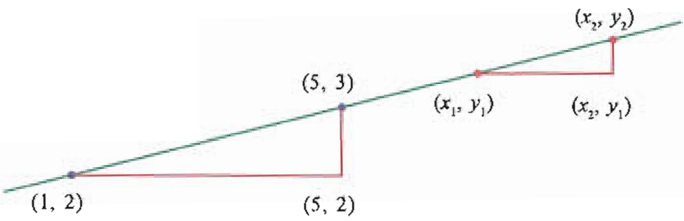

What is the difference between this and the plot of $y = x^2$, seen earlier?

Plot a few more second degree polynomials like this.

We can draw rectangles as in this picture:




Let's focus on the right angled triangles below the line:




Their angles are the same (why?) So, the sides opposite equal angles are proportional. That is,

$$\frac{y_2 - y_1}{3-2} = \frac{x_2 - x_1}{5-1}$$

(How do we get this?) From this, we find

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1}{4}$$

This equation holds wherever on this line we take the points $(x_1, y_1)$ and $(x_2, y_2)$, due to the equality of angles and similarity of triangles as described above.

This means for any pair of points on this line, the difference of $y$-coordinates, divided by the difference of $x$- coordinates is $\frac{1}{4}$ .

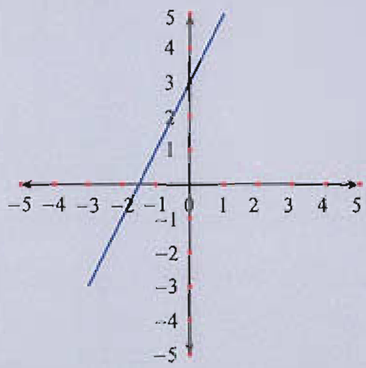
What if we start with some other points instead of (1, 2) and (5, 3) as we did?

For example, let's take (6, 2) and (3, 4). Then for any pair of points on this line, the $y$-difference divided by the $x$-difference would be

$$\frac{2-4}{6-3} = -\frac{2}{3}.$$

### First degree plot

The plot of $y = 2x + 3$ done by a computer is shown below $y$:




Plot some more first degree polynomials. Do you get a straight line every time?

## Slope and rate and proportion

In a line not parallel to the either axis, when we move from one point to another, the $x$-coordinates and the $y$-coordinates change.

For example, consider the line joining (2, 7) and (5, 9). As we move from the first point to the second, the $x$-coordinate increases by 3 and the $y$-coordinate increases by 2. This means that for points on any other position on this line also, as the $x$-coordinate changes by 3 units, the $y$-coordinate changes by 2 units; and we indicate this when we say that the slope of the line is $\frac{2}{3}$.

In other words, at any position on this line, as the $x$-coordinate increases by 1 unit, the $y$-coordinate increases by $\frac{2}{3}$ unit. That is, the number $\frac{2}{3}$ is the rate at which the $y$-coordinate changes with respect to the $x$-coordinate.

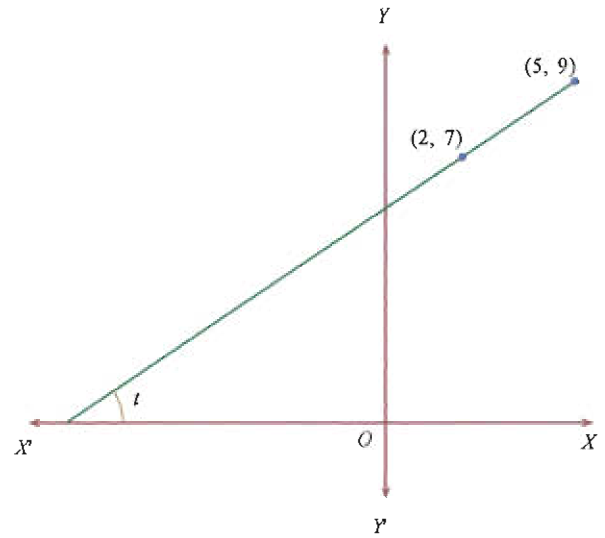
We can put it this way also: for points on this line, the difference in $y$-coordinates is proportional to the difference in $x$-coordinates; and the constant of proportionality is $\frac{2}{3}$.

In general, we have the following:

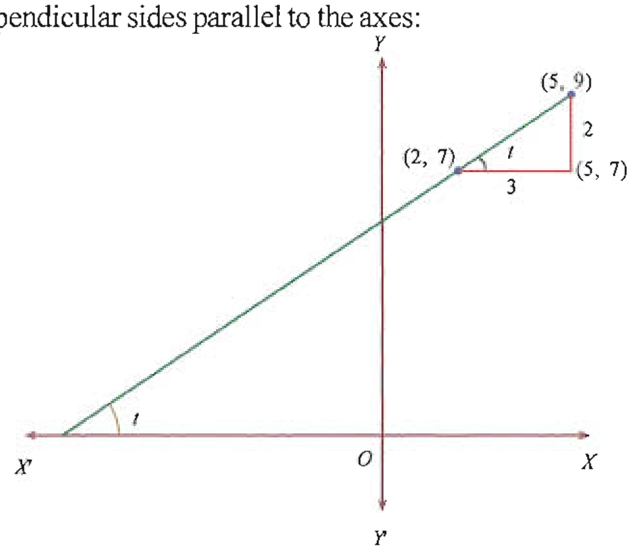
*For any two points on a line not parallel to the y -axis, the difference of the y-coordinates, divided by the difference of the x-coordinates is the same number*

For points on a line parallel to the $x$-axis, this number is zero, right? (Why?) For lines parallel to the $y$-axis, we don't have such a number. (Why?)

Now let's look at this number from another angle. Consider the line joining (2, 7) and (5, 9), for example. Let's take the angle which this line makes with the $x$-axis as $t$.



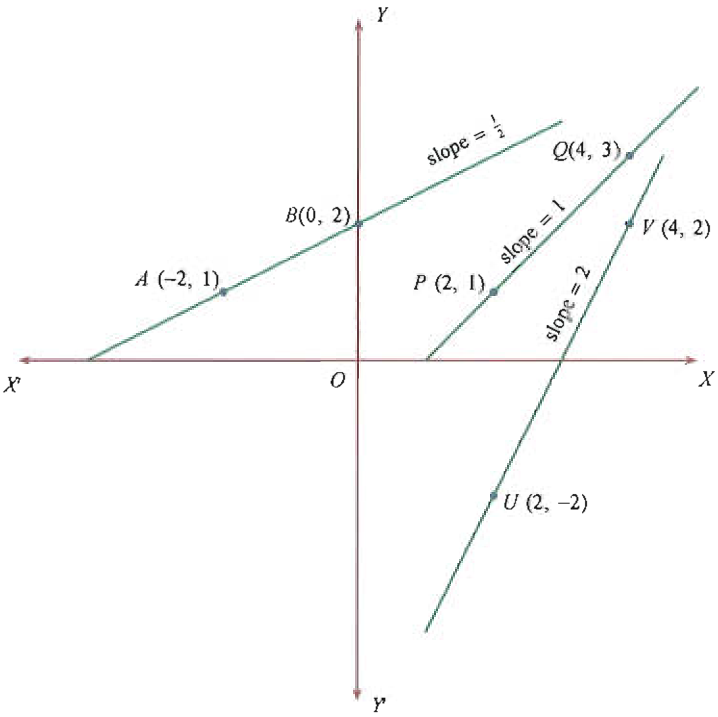

We can draw a small right angled triangle at the top with its perpendicular sides parallel to the axes:



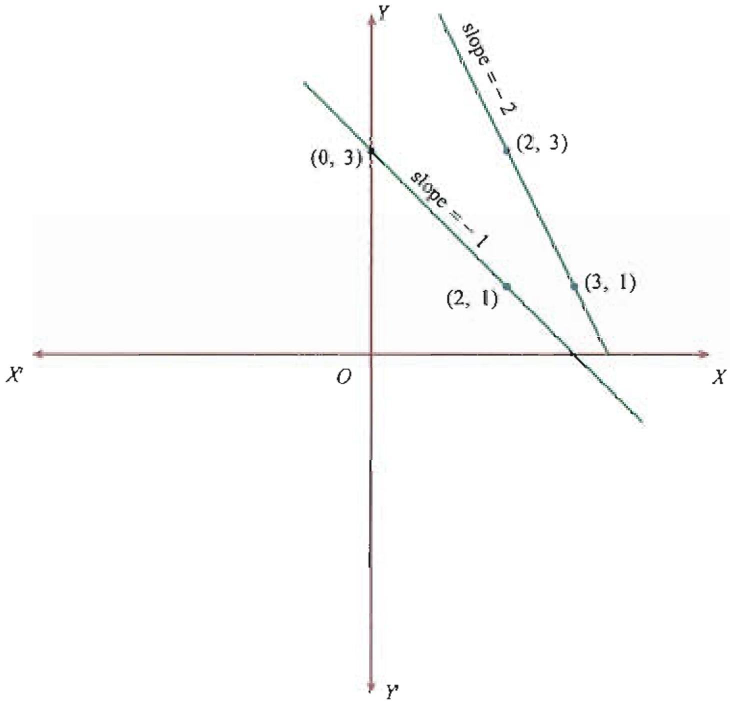

Why is the top angle also $t$? From this right angled triangle, we get

$$\tan t = \frac{2}{3}$$

Thus for a line not parallel to the $y$-axis, if we take a pair of points and divide the difference in $y$-coordinates by the difference in $x$-coordinates, the number we get is the tan measure of the angle which this line makes with the $x$-axis. And this number changes from line to line, as this angle changes. So, this number is called the *slope* of the line.

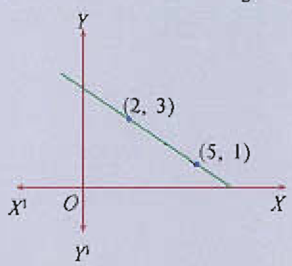



For some lines, the slope is negative, See this picture:




## Negative slopes

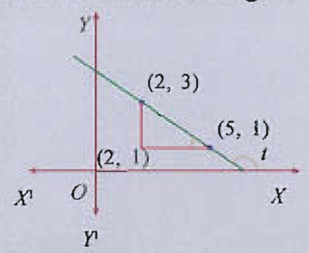
The slopes of some lines are negative. For example, the slope of the line joining (2, 3) and (5, 1) is $-\frac{2}{3}$ right?




This happens, since in this case the $y$-coordinate decreases as the $x$- coordinate increases.

Geometrically, such lines make an angle larger than a right angle with the direction $OX$.

For such lines also, is the slope equal to the tan measure of this angle?




From the pair of similar triangles in the picture above, we get

$$\tan(180 - t) = \frac{2}{3}$$

Since $\tan\ (180 - t) = -\tan t$, by definition, this equation gives

$$-\tan t = \frac{2}{3}$$

and this gives

$$\tan t = -\frac{2}{3}$$

Thus in such cases also, slope gives the tan measure of the angle.

## Physics, algebra and geometry

Suppose a body moves such that the distance travelled is 10 metres in the first second, 15 metres in the next second, 20 metres in the second after and it goes on increasing like this. So, its speed also is increasing every second, as 10m/s during the first second, 15m/s during the next second, 20m/s during the second after that and so on.

In other words, the speed increases by 5m/s every second. In the language of physics, the body has an acceleration of 5 metres per second per second (written 5metres/sec/sec or 5 $m/s^2$)

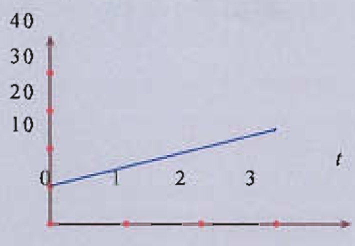
We can use the algebraic equation

$$v = 10 + 5t$$

to calculate the speed $v$ of this body at time $t$.

Now let's mark the various $t$ and $v$ along a pair of perpendicular lines and plot this relation between time and speed. (see the section,

**Relations in physics)**




The slope of this line is 5. Here the slope is the rate at which $v$ changes with respect to $t$; that is, the acceleration.

Let's look at some examples: .

- What is the point at which the line joining (3, 1) and (2,–1) meets the $x$-axis? And the $y$-axis?

  The slope of this line is

  $$\frac{1-(-1)}{3-2}=2$$

  This means for any two points on this line, the $y$- difference divided by the $x$-difference is 2.

  So, if we take the point where this line cuts the $x$-axis as $(x, 0)$, then

  $$\frac{0-1}{x-3}=2$$

  This gives

  $$x - 3 = -\frac{1}{2}$$

  and so

  $$x = \frac{5}{2}$$

  Thus the line cuts the $x$-axis at $\left(\frac{5}{2}, 0\right)$

  Similarly, we can find the point of intersection with the $y$-axis as (0,–5). Try!

  See also if you can do this problem without using algebra.

- Prove that the line joining (3, 5) and (1, 7) passes through the point (5, 3).

  The slope of this line is

  $$\frac{5-7}{3-1}=-1$$

  Now what is the slope of the line joining (3, 5) and (5, 3)?

  $$\frac{5-3}{3-5}=-1$$

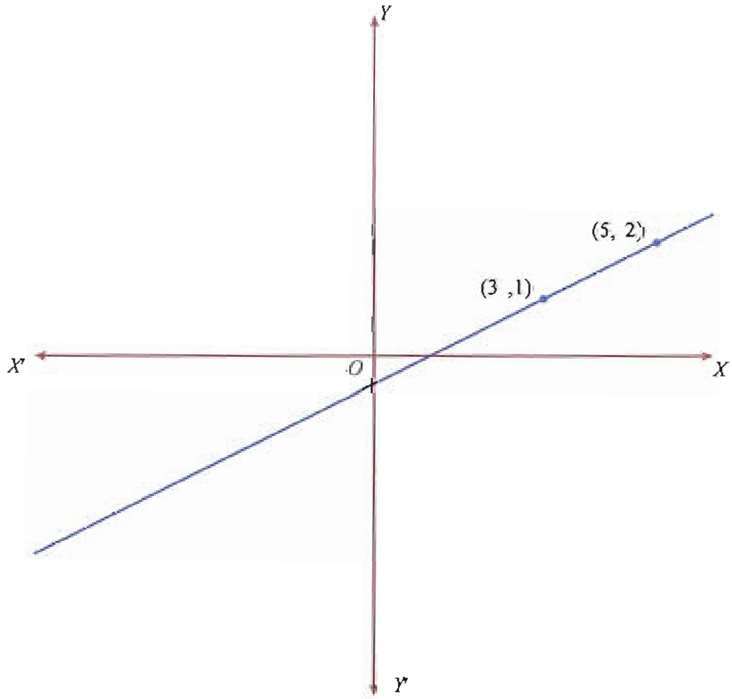
  Since the slopes are equal, these lines make the same angle with the $x$-axis. But they both pass through the point (3, 5). So, they must be the same line.

Now do these problems on your own:

- Does the line joining (2, 3) and (3, −1) pass through the point (5, 6)? What about (5,-9)?
- Prove that the points (1, 4), (4, 1) and $\left(\frac{5}{2}, \frac{5}{2}\right)$ lie on the same line.
- Prove that the points (2, 3), (7, 5), (9, 8), (4, 6) are the vertices of a parallelogram.
- Prove that the line joining the points (2, 1) and (1, 2) and the line joining the points (3, 5) and (4, 7) are not parallel. What are the coordinates of their point of intersection?
- Write down the coordinates of two more points on the line through (1, 3), of slope $\frac{1}{2}$.
- Two lines are drawn through the point (1, 3), one of slope $\frac{1}{2}$ and the other of slope −2. Write the coordinates of one more point on each of these lines. Prove that these lines are perpendicular to each other.

## Equation of a line

The figure below shows the line joining (3, 1) and (5, 2):




Its slope is $\frac{1}{2}$, right?

### Parallel slopes

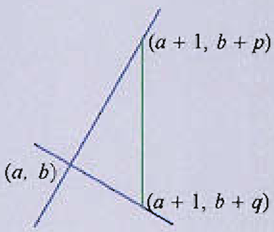
Two lines can have the same slope. For example, the line joining (3, 4), (2, 1) and the line joining (1, 2), (3, 8) are both of slope 3.
If two lines have the same slope, they both make the same angle with the positive direction of the $x$-axis; and so they must be parallel. On the other hand, parallel lines have the same slope (why?)

## Perpendicular slopes

We saw that the slopes of parallel lines are equal. What is the relation between the slopes of perpendicular lines?

Suppose that lines of slopes $p$ and $q$ are perpendicular to each other. Let's take their point of intersection as $(a, b)$.

Then the point $(a + 1, b + p)$ is on the first line and the point $(a + 1, b + q)$ is on the second line (why?)




Since the lines are perpendicular, the points $(a, b)$, $(a + 1, b + p)$, $(a + 1, b + q)$ are the vertices of a right angled triangle. The hypotenuse is the line joining the second and third points. So, the squares of the lengths of the perpendicular sides of this triangle are $p^2 + 1$ and $q^2 + 1$ and the length of the hypotenuse is $|p - q|$. This gives

$$(p^2 + 1) + (q^2 + 1) = (p - q)^2$$

Simplifying, this gives

$$2 = -2pq$$

which means

$$pq = -1$$

Thus, for lines perpendicular to each other, the slope of one is the negative of the reciprocal of the slope of the other.

So, for any point $(x, y)$ on the line, we must have

$$\frac{y-1}{x-3} = \frac{1}{2}$$

From this, we get

$$2(y - 1) = x - 3\ ,$$

and this gives

$$x - 2y - 1 = 0$$

Thus the coordinates $(x, y)$ of any point on this line satisfies the above equation.

Let's think in reverse: suppose we take a pair of numbers $(x, y)$ satisfying the above equation. Would the point with coordinates $(x, y)$ lie on this line?

From the equation $x - 2y - 1 = 0$, we get $x = 2y + 1$. So,

$$\frac{y-1}{x-3} = \frac{y-1}{(2y+1)-3} = \frac{y-1}{2y-2} = \frac{1}{2}$$

Thus the line joining the points $(x, y)$ and $(3, 1)$ and that joining $(3, 1)$ and $(5, 2)$ have the same slope; so they make the same angle with the $x$-axis. And these lines pass through the same point $(3, 1)$. From all these, we see that they are one and the same line. In other words, $(x, y)$ is indeed a point on our line.

Is the point $(3, 4)$ on this line? We have

$$3 - (2 \times 4) - 1 = -6 \neq 0$$

and so this point is not on our line.

How about $(3, 1)$? We have

$$3 - (2 \times 1) - 1 = 0$$

and so this point is on our line.

Let's look at the connection between the line and the equation once more:

- If $(x, y)$ are the coordinates of a point on the line joining $(3, 1)$ and $(5, 2)$, then $x - 2y - 1 = 0$
- If $x, y$ is a pair of numbers with $x - 2y - 1 = 0$, then the point with $(x, y)$ as coordinates is on the line joining $(3, 1)$ and $(5, 2)$.

In short, the collection of the pairs of numbers giving the coordinates of the points on the line joining (3, 1), (5, 2) is the same as the collection of the pairs of numbers satisfying the equation $x-2y-1=0$.

We shorten this by saying

*The equation of the line joining* (3, 1) *and* (5, 2) *is* $x-2y-1=0$

Can you now find the equation of the line joining the points (2, 5) and (−1, 4)?

Let's look at a couple of examples: .

- What is the equation of the line through (2, 5) and of slope $\frac{2}{3}$?

  For any point $(x, y)$ on this line,

$$\frac{y-5}{x-2}=\frac{2}{3}$$

  isn't it? (Why?) And this is the equation of the line. We can simplify this as

$$2x-3y+11=0$$

- What is the slope of the line given by the equation $2x-3y+4=0$?

  Taking the coordinates of two points on this line as $(x_1, y_1)$ and $(x_2, y_2)$, we get

$$2x_1-3y_1+4=0$$

$$2x_2-3y_2+4=0$$

  So, we must have

$$(2x_1-3y_1+4)-(2x_2-3y_2+4)=0$$

  That is,

$$2(x_1-x_2)-3(y_1-y_2)=0$$

  This gives

$$2(x_1-x_2)=3(y_1-y_2)$$

  and so

$$\frac{y_1-y_2}{x_1-x_2}=\frac{2}{3}$$

  Thus the slope of the line is $\frac{2}{3}$.

## Equation of a circle

Suppose a circle is drawn with centre at (1, 2), of radius 4. What is the speciality of the coordinates of the points on it?

The distance from the centre to any point on a circle is equal to the radius, right?

So, if the point $(x, y)$ is on this circle, then

$$(x-1)^2+(y-2)^2=16$$

On the other hand, for any pair of numbers $x$, $y$ satisfying the above equation, the point $(x, y)$ must be on this circle. Thus

*The equation of the circle centred at* (1, 2) *and of radius* 4 *is* $(x-1)^2+(y-2)^2=16$

We can think about this in a slightly different way. First we find two points on this line. For this, we need only find two pairs of numbers satisfying the equation $2x - 3y + 4 = 0$. For example, $(1, 2), (4, 4)$. Since these are points on this line, its slope is

$$\frac{4-2}{4-1} = \frac{2}{3}$$

Now do these problems:

- Prove that for all points on the line joining the origin and the point (4, 2), the $x$-coordinate is double the $y$-coordinate. What is the equation of this line?
- What is the equation of the line joining the points (1, 3) and (2, 7)? Prove that if $(x, y)$ is a point on this line, so is the point $(x + 1, y + 4)$.
- What is the point at which the line $2x + 4y - 1 = 0$ cuts the $x$-axis? What about the $y$-axis?
- Prove that the lines given by the equations $3x + 2y + 5 = 0$ and $3x + 2y - 1 = 0$ are parallel. At what points do they intersect the $x$ - axis? And the $y$-axis?
- At what point do the lines of equations $3x + 2y + 5 = 0$ and $2x - 3y - 1 = 0$ intersect each other? Find one more point on each of these lines. Prove that these lines are perpendicular.

### Mathematical merger

Descartes started the method of studying geometrical figures by converting them to algebraic equations and vice versa, through the device of taking points as pairs of numbers. This united algebra and geometry, which were considered separate branches of mathematics till then. This new geometry is called *Analytic Geometry*.

The branch of mathematics called Calculus, which revolutionized mathematical thought and also other sciences which use mathematics, is based on this new view of geometry. Progress occurs through the synthesis of duals.