9 Polynomials

New equations

Is 7 a factor of 315?

We'll have to divide and see.

$$315 \div 7 = 45$$

So, 7 is a factor of 315.

From the division above, we get

$$315 = 45 \times 7$$

Is 7 a factor of 316?

Division gives the remainder 1; so it is not a factor. We can write

$$316 = (45 \times 7) + 1$$

Now what about dividing the polynomial $x^2 - 1$ by the polynomial $x - 1$?

We know that

$$x^2 - 1 = (x - 1)(x + 1)$$

This means, we can divide $x^2 - 1$ by $x - 1$ without leaving a remainder. In other words

$$\frac{x^2 - 1}{x - 1} = x + 1$$

So, we can say that the polynomial $x - 1$ is a factor of the polynomial $x^2 - 1$.

Similarly $x + 1$ is a factor of $x^2 - 1$.

Now let's see whether $x - 1$ is a factor of $x^2 + 1$.

We can write

$$x^2 + 1 = (x - 1)(x + 1) + 2$$

which means $x^2 + 1$ leaves the remainder 2 on division by $x - 1$.

So, $x - 1$ is not a factor of $x^2 + 1$.

Meaning of factor

The idea of factor which we have seen for natural numbers, can be extended to all integers. For example, since $-12 = 3 \times (-4)$, we can say that -4 is a factor of -12.

What about rational numbers? If we take any two non-zero rational numbers, we can multiply one of these by a suitable rational number to get the other. For example, taking $\frac{2}{3}$ and $\frac{5}{7}$, we can write $\frac{2}{3} = \frac{14}{15} \times \frac{5}{7}$.

(What if one of the numbers is zero?)

So, if we consider the collection of all rational numbers, the idea of factor is not of much use.

In the case of polynomials also, we talk about factors only with respect to the collection of polynomials and not in terms of all algebraic expressions. We have

$$x^2 + 1 = (x - 1)\left(x + 1 + \frac{2}{x - 1}\right)$$

But we don't consider $x - 1$ a factor of $x^2 + 1$ because of this.

Now how do we check whether $x - 1$ is a factor of $x^3 - 1$?

We have to divide and see whether there is a remainder. Since we are dividing by the first degree polynomial $x - 1$, the remainder must be a number. What about the quotient?

As we did in Class 9 let us write

$$x^3 - 1 = (x - 1)(ax^2 + bx + c) + d$$

and find a, b, c, d.

How do we do the multiplication on the right of the above equation? Multiply each term of the first polynomial by each term of the second polynomial and add, right? Thus we get

$$x^3 - 1 = ax^3 + (b - a)x^2 + (c - b)x + (d - c)$$

For this to hold, we need only take

$$\begin{aligned} a &= 1 \\ b - a &= 0 \\ c - b &= 0 \\ d - c &= -1 \end{aligned}$$

That is,

$$a = 1$$

$$b = 1$$

$$c = 1$$

$$d = 0$$

From this we get

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

Since there is no remainder, we can see that $x - 1$ is indeed a factor of $x^3 - 1$.

But then another question arises: is the polynomial $2x - 2$ a factor of $x^3 - 1$?

We can write

$$2x - 2 = 2(x - 1)$$

which gives

$$x - 1 = \frac{1}{2}(2x - 2)$$

Polynomials and numbers

In formulating general principles on polynomials, we often have to include numbers also. For example, the sum of two polynomials may be a number as in

$$(x^2 + x + 1) + (-x^2 - x + 1) = 2$$

Again, the quotient of a polynomial by another may be a number, like

$$\frac{2x+4}{x+2} = 2$$

It is inconvenient to say "polynomial or number" everytime. So, we consider numbers also as polynomials. (We can write $2 = 2x^0$, right?).

All non-zero numbers are said to be zero-degree polynomials. The number 0 itself is treated as a polynomial without degree. This is because, any polynomial multiplied by 0 gives 0 itself so that the general rule, "the degree of a product is the sum of the degrees of the factors" will not hold, whatever number we take as the degree of the zero polynomial.

So, we can rewrite the equation $x^3 - 1 = (x - 1)(x^2 + x + 1)$ as

$$x^3 - 1 = \frac{1}{2}(2x - 2)(x^2 + x + 1)$$

$$= (2x - 2)\,\frac{1}{2}\,(x^2 + x + 1)$$

$$= (2x - 2)\left(\frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{2}\right)$$

What can we say then?

The polynomial $2x - 2$ is also a factor of $x^3 - 1$.

So, what about $3x - 3$?

And $\frac{2}{3}x - \frac{2}{3}$?

What about $1 - x$?

Now in each of the pairs of polynomials given below, check whether the first is a factor of the second:

- $x + 1,\ x^3 - 1$
- $x - 1,\ x^3 + 1$
- $x + 1,\ x^3 + 1$
- $x^2 - 1,\ x^4 - 1$
- $x - 1,\ x^4 - 1$
- $x + 1,\ x^4 - 1$
- $x - 2,\ x^2 - 5x + 1$
- $x + 2,\ x^2 + 5x + 6$
- $\frac{1}{3}x - \frac{2}{3},\ x^2 - 5x + 6$
- $1.3x - 2.6,\ x^2 - 5x + 6$

First degree factors

How do we check whether the polynomial $x - 2$ is a factor of $4x^3 - 3x^2 + x - 1$?

Let's divide and see whether the remainder is zero or not. Since the quotient would be a second degree polynomial and the remainder a number, we write

$$4x^3 - 3x^2 + x - 1 = (x - 2)(ax^2 + bx + c) + d$$

and find a, b, c, d.

Polynomial factors

When we consider numbers also as polynomials, every non-zero number is a factor of every polynomial.

For example,

$$x^2 - 2x + 3 = 2\left(\frac{1}{2}x^2 - x + \frac{3}{2}\right)$$

$$2x^3 + 5x + 7 = \frac{1}{5}\left(10x^3 + 25x + 35\right)$$

and so on.

Moreover, we can multiply any factor of a polynomial by numbers to get new factors. In general, if the polynomial $p(x)$ is a factor of the polynomial $q(x)$, then for any non-zero number a, the polynomial $ap(x)$ is also a factor of $q(x)$.

For checking whether $x - 2$ is a factor, do we need to compute all these? Isn't it enough to find just the remainder?

How do we get d alone from the right side of the above equation? We must make the other terms zero. We know that this equation holds, whatever be the number we take as x.

For example, taking $x = 1$, this equation gives,

$$(4 \times 1^3) - (3 \times 1^2) + 1 - 1 = (1 - 2)(a \times 1^2 + b \times 1 + c) + d$$

and reading this in reverse, we get

$$-(a + b + c) + d = 1$$

What if we take $x = 2$?

$$(4 \times 2^3) - (3 \times 2^2) + 2 - 1 = (2 - 2)((a \times 2^2) + (b \times 2) + c) + d$$

which means

$$0 \times (4a + 2b + c) + d = 21$$

That is, $d = 21$

What does this mean? On dividing the poynomial $4x^3 - 3x^2 + x - 1$ by $x - 2$, we get the remainder 21; and so $x - 2$ is not a factor of $4x^3 - 3x^2 + x - 1$.

Let's now check, whether the first degree polynomial $x - 3$ is a factor of $2x^3 - 5x^2 - 4x + 3$.

Since we are not interested in the quotient, let's write it in short as simply $q(x)$; and the remainder as r. Thus

$$2x^3 - 5x^2 - 4x + 3 = (x - 3)\, q(x) + r$$

Now we need only check whether r is zero. To get r, all we need to do is to take $x = 3$ in the above equation, right? This gives

$$(2 \times 3^3) - (5 \times 3^2) - (4 \times 3) + 3 = (3 - 3)\, q(3) + r$$

From this, we get

$$0 \times q(3) + r = 54 - 45 - 12 + 3$$

That is, $r = 0$

What does this mean?

Meaningful math

We have noted that the various kinds of numbers, natural numbers, rational numbers, irrational numbers, were created to indicate various types of measures; and that the very instances where these numbers are used, determine the way these numbers are operated upon.

In trying to divide 14 sweets equally among 3 children, we end up with 2 sweets which cannot be given whole; and in trying to cut a 14 metre long string into 3 metre pieces, we get a 2 metre piece which is not of the required length. These are some of the instances leading to the mathematical idea that 14 divided by 3 leaves the remainder 2.

Now consider this question:

What is the remainder on dividing -14 by -3?

In view of the above remarks, what is the meaning of this question?

The polynomial $x-3$ is a factor of $2x^3-5x^2-4x+3$.

Let's see how we can write this as a general principle. We want to check whether the first degree polynomial $x-a$ is a factor of the polynomial $p(x)$.

We write the quotient polynomial on dividing $p(x)$ by $x-a$ as $q(x)$ and the remainder as r. Then we get the identity

$$p(x)=(x-a)q(x)+r$$

This holds for all numbers x. In particular, if we take $x=a$, then this gives.

$$p(a)=(a-a)q(a)+r$$

and this means

$$p(a)=r$$

Thus we have this general result:

The remainder on dividing the polynomial $p(x)$ by the polynomial $x-a$ is $p(a)$.

Now what happens if $p(a)=0$? This means the remainder on dividing $p(x)$ by $x-a$ is zero; that is, $x-a$ is a factor of $p(x)$. On the other hand, what if $p(a)\neq 0$? Since the remainder is not zero, $x-a$ is not a factor of $p(x)$.

For the polynomial $p(x)$, and for the number a, if we have $p(a)=0$, then $x-a$ is a factor of $p(x)$; if $p(a)\neq 0$, then the polynomial $x-a$ is not a factor of $p(x)$.

The first principle is called the *Remainder Theorem* and the second is called the *Factor Theorem*.

Let's look at some examples:

- Is the polynomial $x-2$ a factor of the polynomial $x^4-x^3-x^2-x-2$?

  The Factor Theorem says that to check this, we only need to put $x=2$ in $x^4-x^3-x^2-x-2$ and check whether we get zero.

  $$2^4-2^3-2^2-2-2=16-8-4-2-2=0$$

  So, $x-2$ is indeed a factor of $x^4-x^3-x^2-x-2$.

Meaning of remainder

To extend the idea of remainder to all integers, we must first interpret this idea *in purely mathematical terms* for natural numbers.

We say that when the natural number a is divided by the natural number b, the quotient is q and remainder is r, if q and r satisfy the following conditions:

1. $a=qb+r$
2. q and r are natural numbers or zero
3. $r<b$

We can extend this definition to all integers with some minor modifications:

We say that when the integer a is divided by the integer b, the quotient is q and remainder is r, if q and r satisfy the following conditions:

1. $a=qb+r$
2. q and r are integers
3. $r=0$ or $0<r<|b|$

For example, taking the numbers -14 and -3, we find

1. $-14=5\times(-3)+1$
2. 5 and 1 are integers
3. $0<1<|-3|$

So, we say that on dividing -14 by -3, the quotient is 5 and the remainder is 1.

Polynomial division

Once we include numbers also as polynomials, we can extend to polynomials, the definition of quotient and remainder for integers to in much the same way.

We say that when the the polynomial $a(x)$ is divided by the polynomial $b(x)$, the quotient is $q(x)$ and remainder is $r(x)$, if $q(x)$ and $r(x)$ satisfy the following conditions:

1. $a(x) = q(x)b(x) + r(x)$
2. $q(x)$ and $r(x)$ are polynomials.
3. $r(x) = 0$ or deg $r(x) <$ deg $b(x)$

In this, we denote by deg, the degree of a polynomial.

- Is $x + 3$ a factor of $2x^2 + 3x - 5$?

  The Factor Theorem talks about factors of the type $x - a$; but here what we want to check is $x + 3$.

  Can't we write $x + 3$ in this form also?

$$x + 3 = x - (-3)$$

  So, we need only check whether $x = -3$ in $2x^2 + 3x - 5$ gives zero:

$$(2 \times (-3)^2) + (3 \times (-3)) - 5 = 18 - 9 - 5 = 4$$

  Since we don't get zero, we find that the polynomial $x + 3$ is not a factor of $2x^2 + 3x - 5$.

- Is the polynomial $2x - 3$ a factor of the polynomial $2x^2 - x - 3$?

  How do we rewrite $2x - 3$ in a form suitable for the application of the Factor Theorem?

$$2x - 3 = 2\left(x - \frac{3}{2}\right)$$

  Now we check whether the polynomial $x - \frac{3}{2}$ is a factor of $2x^2 - x - 3$ (Would that do?)

  For this, we take $x = \frac{3}{2}$ in $2x^2 - x - 3$ and find

$$2 \times \left(\frac{3}{2}\right)^2 - \frac{3}{2} - 3 = \left(2 \times \frac{9}{4}\right) - \frac{3}{2} - 3 = \frac{9}{2} - \frac{3}{2} - 3 = 0$$

  Thus $x - \frac{3}{2}$ is a factor of $2x^2 - x - 3$ and so $2x - 3$ is also a factor of $2x^2 - x - 3$ (why?)

Now try these problems on your own:

- Check whether each of the polynomials listed below is a factor of $3x^3 - 2x^2 - 3x + 2$; if not, find the remainder.

  - $x - 1$ • $3x - 2$ • $2x - 3$

- $x + 1$ • $3x + 2$ • $2x + 3$

- What is the remainder on dividing the polynomial $p(x)$ by $ax + b$? What is the condition under which $ax + b$ is a factor of the polynomial $p(x)$?
- Is $x - 1$ a factor of $x^{100} - 1$? What about $x + 1$?
- Is $x - 1$ a factor of $x^{101} - 1$? What about $x + 1$?
- Prove that $x - 1$ is a factor of $x^n - 1$ for every natural number n.
- Prove that $x + 1$ is a factor of $x^n - 1$ for every even number n.
- Prove that $x + 1$ is not a factor of $x^n - 1$ for every odd number n.
- What number added to $3x^3 - 2x^2 + 5x$ gives a polynomial for which $x - 1$ is a factor?
- What first degree polynomial added to $3x^3 - 2x^2$ gives a polynomial for which both $x - 1$ and $x + 1$ are factors?

Factorization

How do we find the factors of the polynomial, $x^2 + x - 12$?

It is easy to check whether a given polynomial such as $x - 2$ or $2x + 1$ is a factor of $x^2 + x - 12$. Instead, how do we directly find a factor of $x^2 + x - 12$?

According to the Factor Theorem, to find the first degree factors of $x^2 + x - 12$, we need only find those numbers x which make $x^2 + x - 12$ zero.

In other words, we need only solve the second degree equation

$$x^2 + x - 12 = 0$$

That we know:

$$x = \frac{-1 \pm \sqrt{1+48}}{2} = \frac{-1 \pm 7}{2} = 3 \text{ or } -4$$

Sum of powers

We noted that $x - 1$ is a factor of $x^n - 1$, for every natural number n. What is the quotient here?

We have seen that

when $n = 2$, $\frac{x^2-1}{x-1} = x + 1$

and when $n = 3$, $\frac{x^3-1}{x-1} = x^2 + x + 1$

In the same manner, it's not difficult to see that

$$\frac{x^4-1}{x-1} = x^3 + x^2 + x + 1$$

In general, for any natural number n,

$$\frac{x^n-1}{x-1} = x^{n-1} + x^{n-2} + \ldots\ldots + x^2 + x + 1$$

Reading this in reverse,

$$1 + x + x^2 + \ldots\ldots\ldots + x^{n-1} = \frac{x^n-1}{x-1}$$

This equation is true for all numbers except 1. For example, taking $x = 2$ in this,

$$1 + 2 + 2^2 + \ldots\ldots + 2^{n-1}$$

$$= \frac{2^n-1}{2-1} = 2^n - 1$$

as we have seen in the section **Growing triangles** of the lesson **Arithmetic Sequences.** Similarly

$$1 + 3 + 3^2 + \ldots\ldots + 3^{n-1}$$

$$= \frac{3^n-1}{3-1} = \frac{3^n-1}{2}$$

and

$$1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots + \frac{1}{2^{n-1}}$$

$$= \frac{\frac{1}{2^n} - 1}{\frac{1}{2} - 1} = 2\left(1 - \frac{1}{2^n}\right)$$

Thus if we take $x = 3$ or if we take $x = -4$, we can make $x^2 + x - 12$ zero. So, by the Factor Theorem, $x - 3$ and $x - (-4) = x + 4$ are factors of the polynomial $x^2 + x - 12$.

If we multiply these two factors, we get

$$(x - 3)(x + 4) = x^2 + x - 12$$

which is the polynomial we started with.

How do we find the factors of $3x^2 + 5x + 2$ like this?

As before, we first solve the equation

$$3x^2 + 5x + 2 = 0$$

We find

$$x = \frac{-5 \pm \sqrt{25-24}}{6} = \frac{-5 \pm 1}{6} = -\frac{2}{3} \text{ or } -1$$

Again, as in the first problem, using the Factor Theorem we see that $x + \frac{2}{3}$ and $x + 1$ are factors of $3x^2 + 5x + 2$.

$$\left(x+\frac{2}{3}\right)(x + 1) = x^2 + \frac{5}{3}x + \frac{2}{3}$$

This is not the original polynomial $3x^2 + 5x + 2$ we started with. However, we can write

$$x^2 + \frac{5}{3}x + \frac{2}{3} = \frac{1}{3}(3x^2 + 5x + 2)$$

Thus

$$\left(x+\frac{2}{3}\right)(x + 1) = \frac{1}{3}(3x^2 + 5x + 2)$$

From this we find

$$3x^2 + 5x + 2 = 3\left(x+\frac{2}{3}\right)(x + 1) = (3x + 2)(x + 1)$$

Next, let's see how $6x^2 - 7x - 3$ is split into factors.

Another way

Some polynomials of the form $x^2 + ax + b$ can be easily factorized. Take $x^2 + 5x + 6$, for example. If we take its factors as $x + a$ and $x + b$, then we have

$$x^2 + 5x + 6 = (x + a)(x + b)$$
$$= x^2 + (a + b)x + ab$$

For this to hold, we need only have

$$a + b = 5$$
$$ab = 6$$

In other words, we want to find two numbers with sum 5 and product 6. A moment's thought will give these as 2 and 3. Thus

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

See if you can factorize $x^2 + 10x + 24$ like this.

What about $x^2 - 10x + 24$?

First we solve

$$6x^2 - 7x - 3 = 0$$

(Why?)

This gives

$$x = \frac{7 \pm \sqrt{49+72}}{12} = \frac{7 \pm 11}{12} = \frac{3}{2} \text{ or } -\frac{1}{3}$$

Next we find the product of $x - \frac{3}{2}$ and $x + \frac{1}{3}$

$$\left(x - \frac{3}{2}\right)\left(x + \frac{1}{3}\right) = x^2 + \left(\frac{1}{3} - \frac{3}{2}\right)x - \left(\frac{3}{2} \times \frac{1}{3}\right)$$

$$= x^2 - \frac{7}{6}x - \frac{1}{2}$$

$$= \frac{1}{6}(6x^2 - 7x - 3)$$

Now we need only write this in reverse:

$$6x^2 - 7x - 3 = 6\left(x - \frac{3}{2}\right)\left(x + \frac{1}{3}\right)$$

$$= 2\left(x - \frac{3}{2}\right) \times 3\left(x + \frac{1}{3}\right)$$

$$= (2x - 3)(3x + 1)$$

Let's look at one more example. How do we factorize $x^2 - 2x - 1$?

Solving the equation

$$x^2 - 2x - 1 = 0$$

we get

$$x = 1 \pm \sqrt{2}$$

In other words, the solutions of this equation are $1 + \sqrt{2}$ and $1 - \sqrt{2}$.

Let's multiply the polynomials formed by subtracting each of these from x:

$$(x - (1 + \sqrt{2}))(x - (1 - \sqrt{2}))$$

$$= x^2 - \left((1 + \sqrt{2}) + (1 - \sqrt{2})\right)x + (1 + \sqrt{2})(1 - \sqrt{2})$$

$$= x^2 - 2x + \left(1^2 - (\sqrt{2})^2\right)$$

$$= x^2 - 2x - 1$$

Factorization and solution

We saw that to factorize a polynomial $p(x)$, we need only solve the equation $p(x) = 0$. On the other hand, if we are able to factorize a polynomial $p(x)$, then we get the solutions of the equation $p(x) = 0$ as well.

For example, look at the equation

$$x^2 + 5x + 6 = 0$$

Once we find

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

we can write the equation we started with as

$$(x + 2)(x + 3) = 0$$

For this to hold, we must find numbers x such that the product of the numbers $x + 2$ and $x + 3$ is zero.

For a product to be zero, we need only have one of the factors zero. Thus we need only find x such that either $x + 2$ or $x + 3$ is zero. That is,

$$x + 2 = 0 \text{ or } x + 3 = 0$$

which gives

$$x = -2 \text{ or } x = -3$$

Third degree polynomials

Consider the polynomial $x^3-6x^2+11x-6$? How do we factorize it? To use the Factor Theorem, we must solve the equation

$$x^3 - 6x^2 + 11x - 6 = 0$$

But we haven't seen any general technique for doing this.

We can check some possibilities. If we take $x = 1$ in this polynomial, we get $1 - 6 + 11 - 6 = 0$. So, $x - 1$ is a factor. How do we find the other factors?

If we divide $x^3 - 6x^2 + 11x - 6$ by $x - 1$, we get $x^2 - 5x + 6$ (try!) Also, for $x^2 - 5x + 6 = 0$, we must have $x = 2$ or $x = 3$. So, what do we have now?

$$x^3 - 6x^2 + 11x - 6$$
$$= (x - 1)(x^2 - 5x + 6)$$
$$= (x - 1)(x - 2)(x - 3)$$

Can you factorize $x^3 - 4x^2 + x + 6$ like this?

Thus

$$x^2 - 2x - 1 = (x - 1 - \sqrt{2})(x - 1 + \sqrt{2})$$

Can we factorize all polynomials like this?

Take the polynomial $x^2 + 1 = 0$, for example. If it has first degree factors, then the equation $x^2 + 1 = 0$ must have solutions; since it has none (why?), this polynomial does not have first degree factors.

Now try these problems:

- Write each of the polynomials listed below as a product of two first degree polynomials:
  - $2x^2 + 5x + 3$
  - $x^2 + 2x - 1$
  - $x^2 + 3x + 2$
  - $x^2 - 2$
  - $4x^2 + 20x + 25$
  - $x^2 - x - 1$
- Prove that none of the polynomials listed below has first degree factors:
  - $x^2 + x + 1$
  - $x^4 + 1$
  - $x^2 - x + 1$
  - $x^4 + x^2 + 1$

Project

- Find separately the speciality of the coefficients of polynomials for which $x - 1$, $x + 1$ or $x^2 - 1$ is a factor.