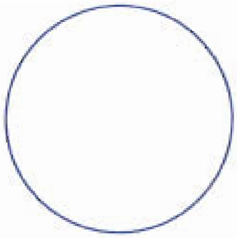
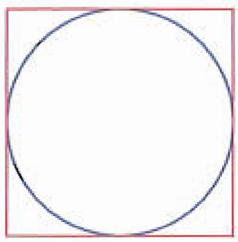
8 Tangents

Around a circle

Draw a circle of any radius you please:

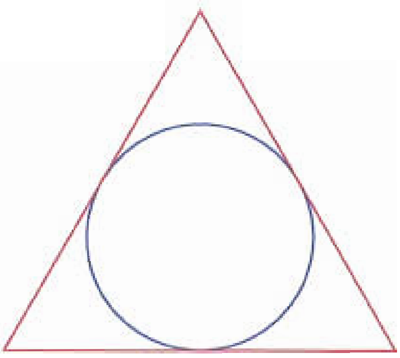


Now draw a square around it as shown below.



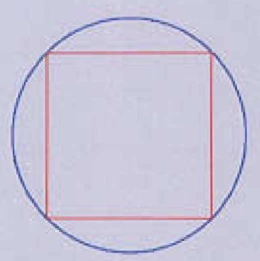
How did you draw the sides?

Next draw a circle and this time, draw an equilateral triangle around it as shown below.

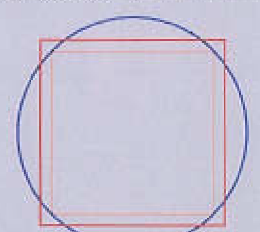


Growing square

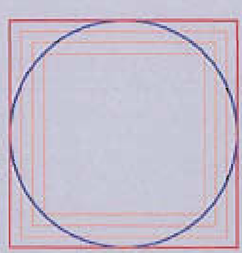
It is not difficult to draw a square inside a circle like this, is it?



We can slightly increase the lengths of the sides and draw like this:



If we keep on increasing the lengths of the sides little by little, we get a square like this:



Can we push out any square within the circle in this manner?

Not that easy, is it?

In the picture of the square and in the picture of the triangle, each side passes through how many points of the circle?

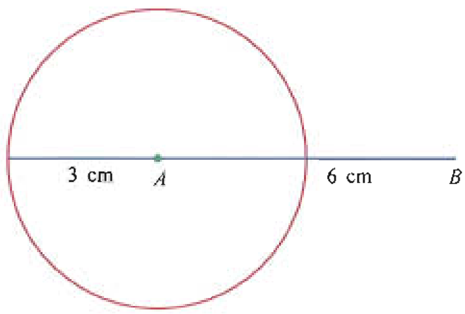
Let's look at such relations between lines and circles in detail.

Lines and circles

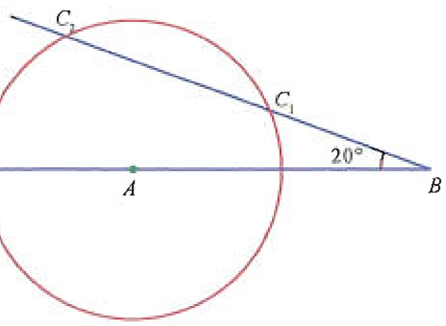
Have you seen before, instances where a line passes through a single point on a circle?

Look at this example. We want to draw triangle ABC with AB of length 6 centimetres, AC is of length 3 centimetres and the angle at B of $20°$. (Do you remember a similar problem done in Class 8?)

We start by drawing AB of length 6 centimetres. We want C to be 3 centimetres away from A; this means C must be a point on the circle of radius 3, centred at A.

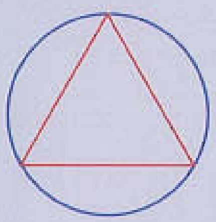


What next? Since the angle at B is to be $20°$, let's draw a line of this slant through B:



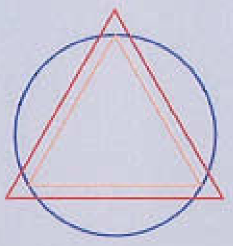
Growing triangles

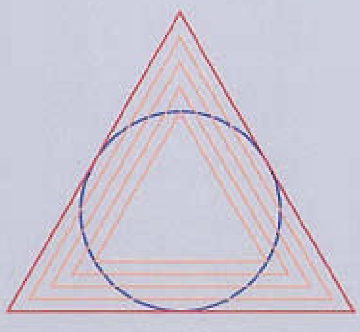
Can you draw an equilateral triangle within a circle, as shown below?



(Recall the section **Arcs, angles and chords**, of the lesson **Circles**)

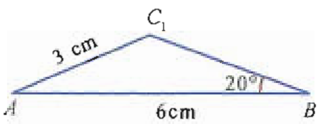
We can enlarge this triangle little by little, as in the case of a square:

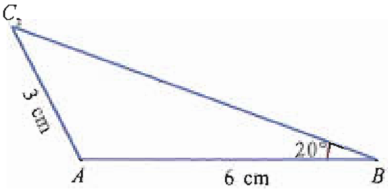




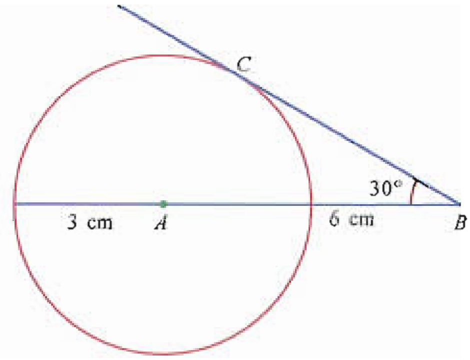
How long should we make the sides to get the triangle outside?

Thus we get two triangles with the given specifications:

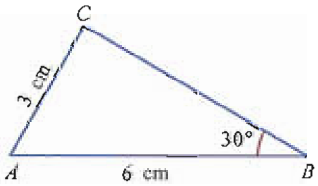




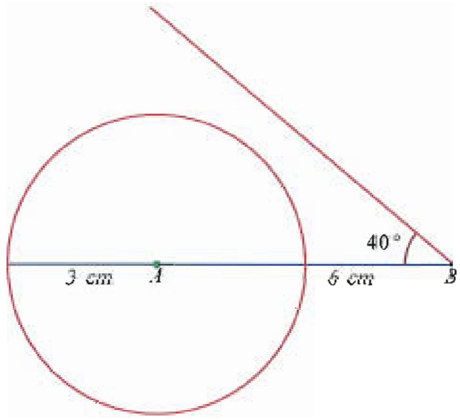
Now suppose we want the angle at B to be 30°.



We get only one triangle:



And if we make it 40°?

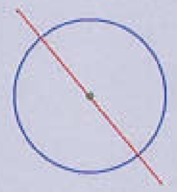


We see the 20° line cutting the circle at two points; the 40° line has nothing to do with the circle.

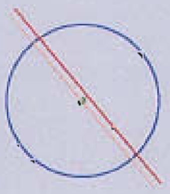
What about the 30° line? It just touches the circle. Such a line is called a *tangent* to the circle.

Sliding line

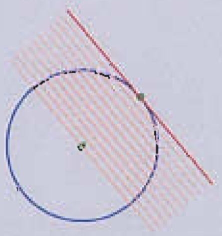
See this picture:



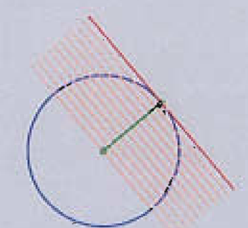
A circle and a line through its centre. Suppose we slide the line up a bit:



As we go on sliding the line slowly, we get a line which passes through a single point of the circle, right?

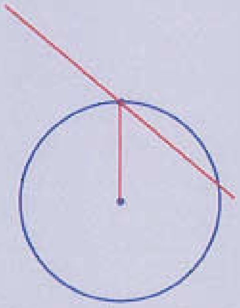


And the line joining the centre and this final point is perpendicular to all these lines:



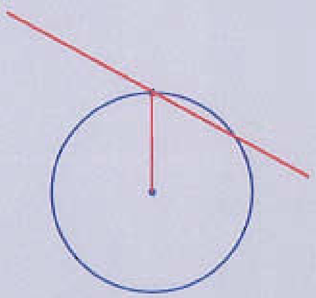
Rotating line

Look at this picture:

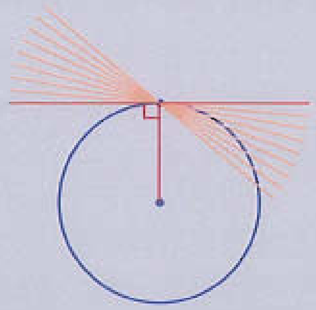


A circle, a radius and a slanted line through its end.

What if we rotate this line a bit, about the top point?

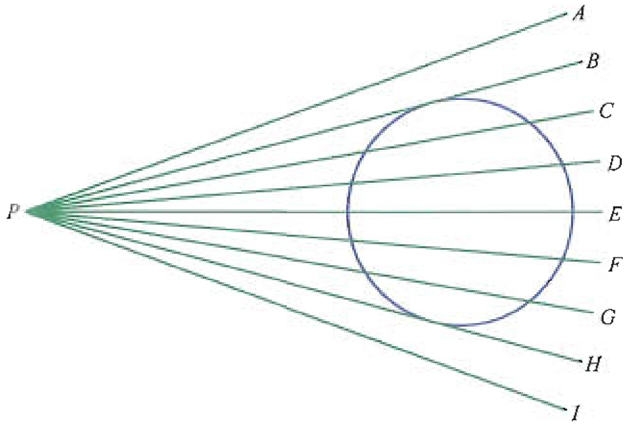


As we go on rotating slowly, we get to a stage when the line is perpendicular to the radius, don't we?



This line goes through how many points of the circle?

Now look at this picture:

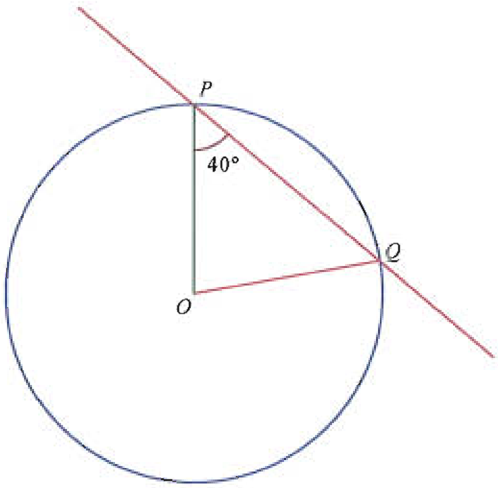


Among all the lines, only two are tangents to the circle. Which are they?

Now take a look at the triangles we drew earlier. In the case of two triangles, the top angle of one of these is greater than a right angle and in the other, it is less than a right angle. Is there any relation between these two angles? See how we got the top vertices of these triangles.

What is the top angle, in the case of a single triangle?

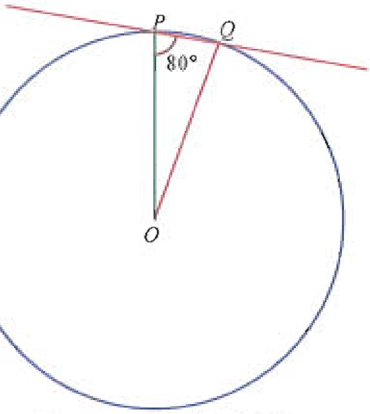
Let's draw another picture.



How much is $\angle OQP$ in this?

Draw some more pictures like this with the angle at P increased to 50°, 60° and so on, by shifting the position of Q. What do you see?

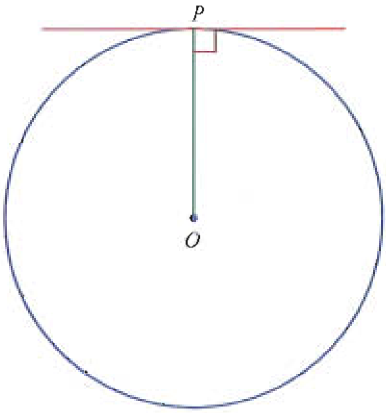
As the angle at P increases, the point Q gets closer to P; and ΔPOQ becomes thinner.



What happens when the angle at P is 90°?

Would this line meet the circle at any other point? If so, the angle at Q would also have to be 90°. How can there be two right angles in a triangle?

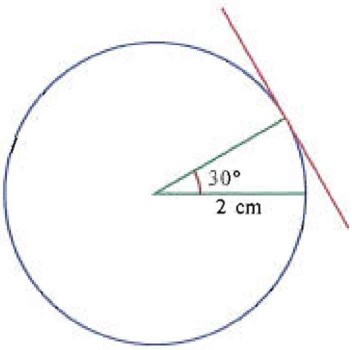
So, there is no other point common to this line and the circle; that is, it is a tangent to the circle.



What general principle do we get from this?

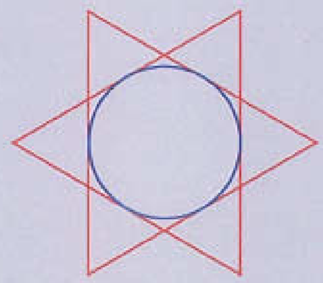
> *A line drawn through any point of a circle, perpendicular to the radius through that point, is a tangent to the circle.*

Now try drawing the pictures below according to the given specifications.

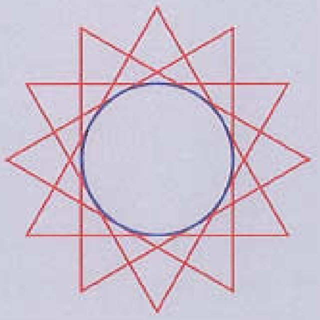


Circle from lines

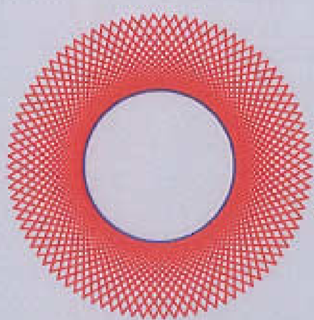
See this picture of a star made by six tangents to a circle:

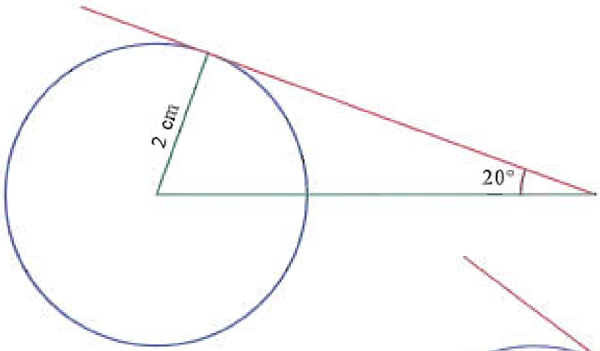


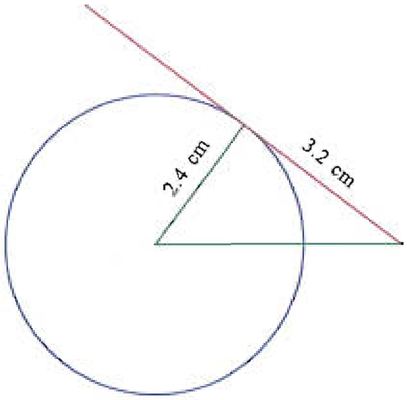
We can increase the number of tangents to 12:



And this is a picture drawn by a computer, using 90 tangents:







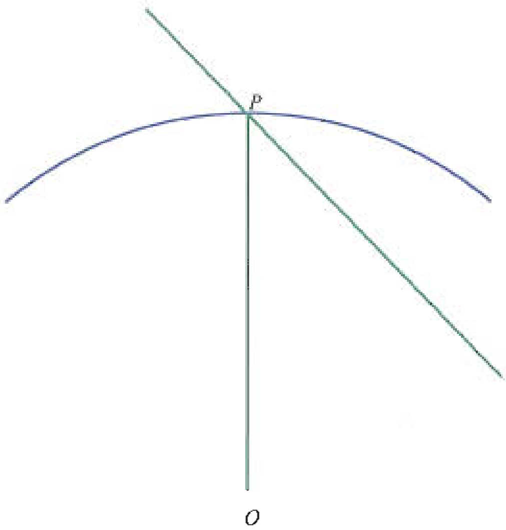
Draw a diameter AB in a circle. Prove that the tangets at A and B do not intersect.

Theory and application

We saw that we can draw tangents by drawing perpendiculars to radius. Are all tangents like this? In other words, is every tangent perpendicular to the radius through the point of contact?

To answer this, first draw a circle and a radius, and then draw a line through the end of the radius, not perpendicular to the radius. You can see that it cuts the circle at another point. What is the position of this second point? Can you specify it without seeing the whole circle?

See this picture:

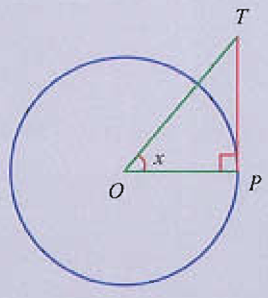


Name and meaning

The word tangent comes from the Latin root *tangere*, meaning to touch.

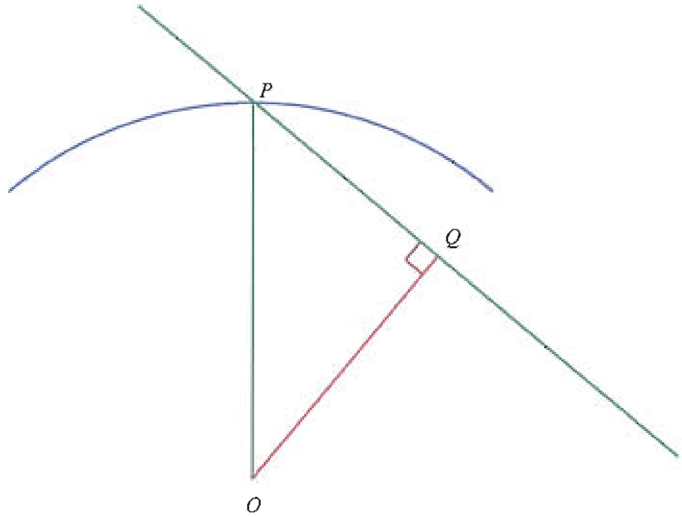
The tan measure used in trigonometry is also an abbreviation of the word tangent, right? What is the connection between this measure of an angle and a line touching a circle?

See this picture:

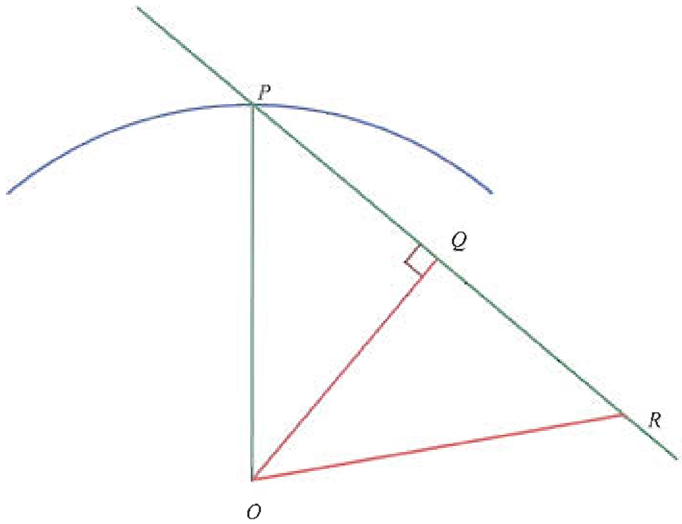


If we take the radius of the circle as 1, then the length of the tangent PT is indeed $\tan x$, isn't it?

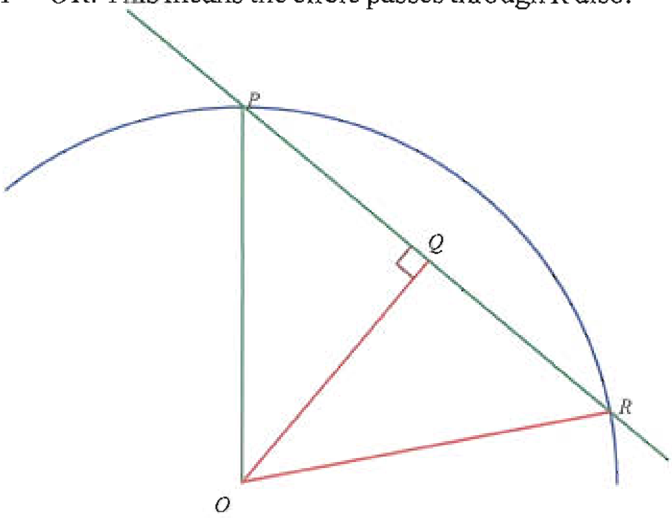
The line through P is not perpendicular to the radius OP; so we can draw a perpendicular to this line from O.



Now mark R ahead of Q at the same distance from P.

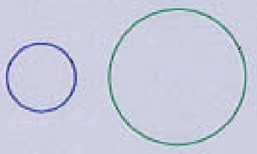


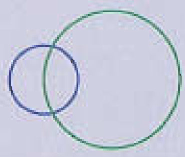
Now the triangles OPQ and ORQ are congruent. (Why?) So, $OP = OR$. This means the circle passes through R also.



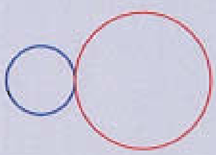
Touching circles

Like a circle and a line, two circles may not intersect, or intersect at two points:

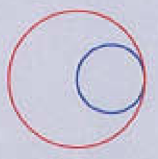




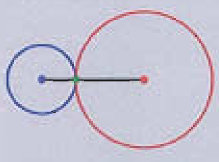
It may happen that two circles only touch:

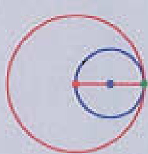


Instead of tocuhing *externally* as in the picture above, two circles may touch *internally* like this:



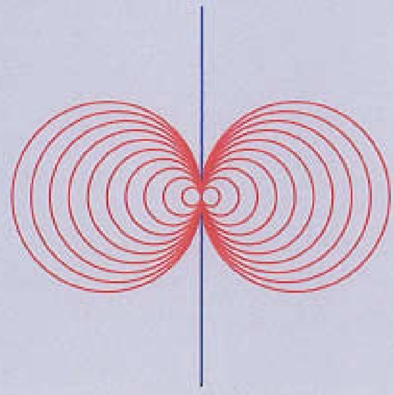
Euclid has proved that, however they touch, the point of contact and the centres of the circles are on the same line:



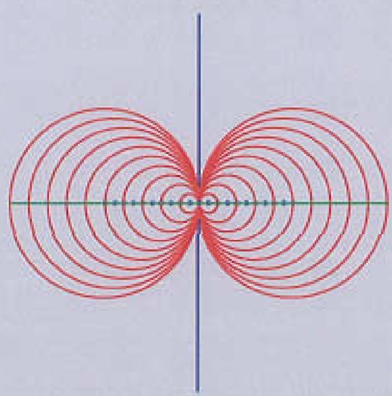


Bunch of circles

There's only one tangent at a specific point on a circle. But there are several circles touching a line at a specific point. See this picture:



These circles all touch one another. So, their centres are all on the same line, And the common tangent is perpendicular to this line.



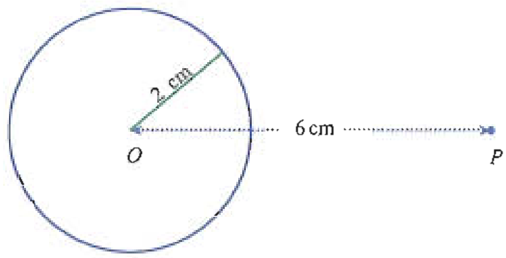
What did we see here? If a line through P is not perpendicular to the radius OP, then it cuts the circle at another point also; on the other hand, the tangent at P does not meet the circle at any other point. So, the tangent at P has to be perpendicular to the radius OP.

Let's write this as a general principle:

Any tangent to a circle is perpendicular to the radius through the point of contact.

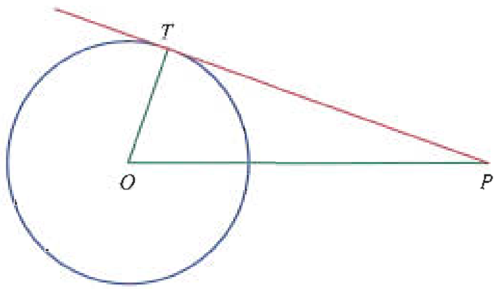
Let's look at an application of this.

Draw a circle of radius 2 centimetres and mark a point 6 centimetres away from its centre.



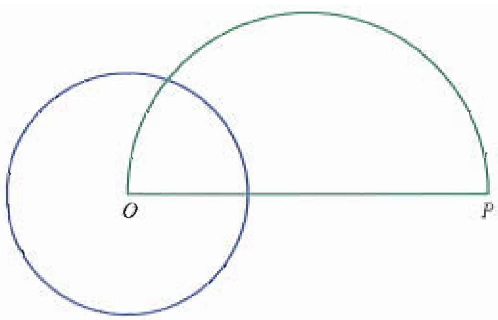
How do we draw a tangent to the circle, passing through this point?

First let's draw a rough sketch to see how we go about doing this:

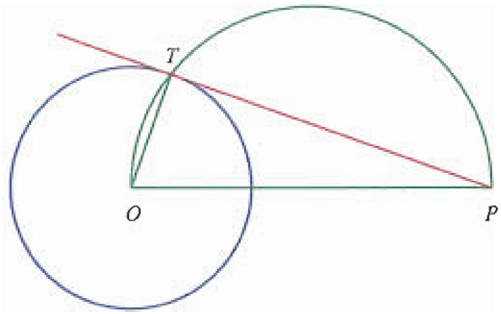


Since what we want is a tangent, the angle at the top should be a right angle. So, what we want is a right angled triangle with the bottom line as hypotenuse. But this can be done by drawing a semicircle, right? (Surely, you haven't forgotten what you have seen in the lesson on circles?)

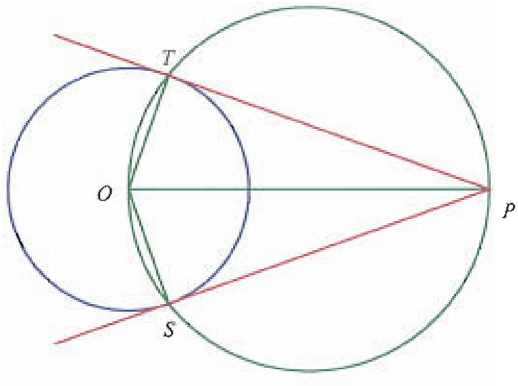
So, let's get down to the actual business of drawing.



Joining any point on this semicircle with O and P, we get a right angled triangle with OP as the hypotenuse. But in the right angled triangle we need, the third vertex should be a point on the small circle also. So, we take the point of intersection of this circle and the newly drawn semicircle.



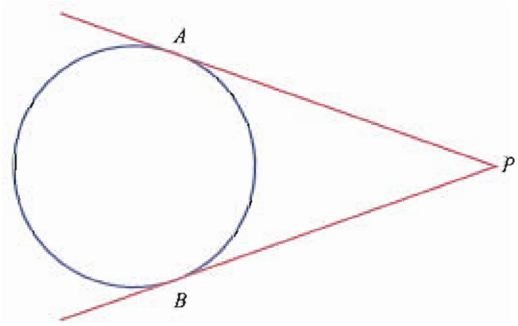
Our job is done, once we draw the line joining P and T. But we can think about another point—won't we also get a tangent by drawing a semicircle downwards?



So, from P, we can draw two tangents.

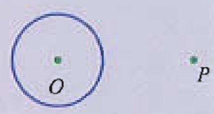
Not only this, but we can also see from the figure above that the lines PT and PS are equal. These we can call the *length of the tangents from P*. So, the lengths of the tangents to a circle from a point outside are equal. How do we prove this?

In the figure below, the lengths of the tangents from P to the circle are PA and PB.

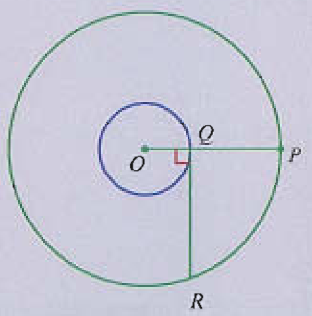


Another method

Euclid uses another method to draw tangents to a circle from an external point:

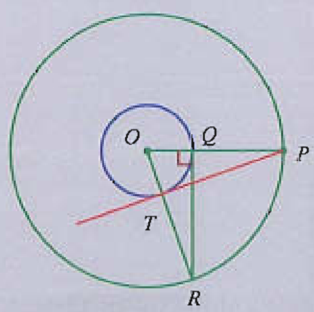


Join OP and draw another circle with this length as radius, centred at O. Draw the perpendicular to OP from the point where it cuts the original circle, and extend it to meet the second circle:



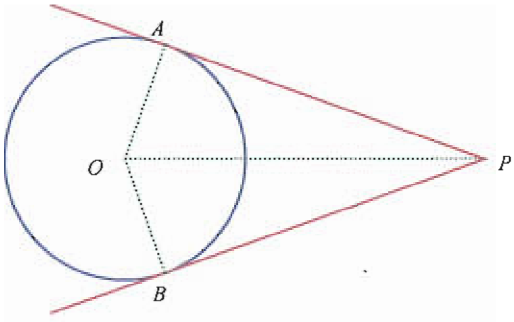
Join OR and join the point where it cuts the original circle with P.

This gives the tangent:



Can you prove it is so? Can you draw the other tangent from P likewise?

We want to prove that $PA = PB$. For this, join P, A, B to the center O of the circle.



The line AP is a tangent at the point A on the circle, and the line OA is the radius through A, so that $\angle OAP = 90°$.

Thus ΔOAP is a right angled triangle and so by Pythagoras Theorem,

$$PA = \sqrt{OP^2 - OA^2}$$

Likewise, since BP is the tangent to the circle at B and BO is the radius through B, we have $\angle OBP = 90°$ and so from the right angled triangle OBP, we get

$$PB = \sqrt{OP^2 - OB^2}$$

Now since OA and OB are radii of the circle, we have

$$OA = OB$$

From the three equations above, we get

$$PA = \sqrt{OP^2 - OA^2} = \sqrt{OP^2 - OB^2} = PB$$

Let's write this as a general principle:

From any point outside a circle, we can draw two tangents; and the lengths of these tangents are equal.

Now try your hand at these problems:

- Circles centred at A and B cut at P. Prove that if AP is a tangent to the circle centred at B, then BP is tangent to the circle centred at A.

How to write a proof

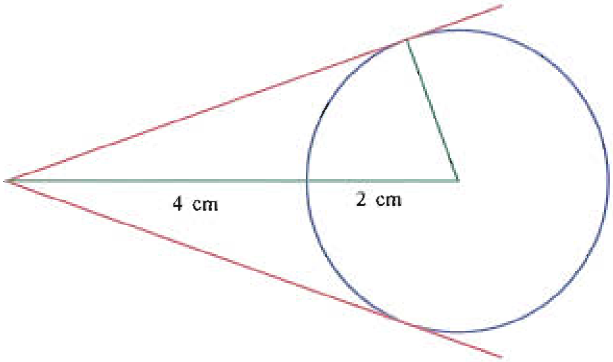
We have said much about the master geometer Euclid and his work named Elements. (See the section **Circles and triangles,** of the lesson **Math Drawing** in the Class 7 textbook.)

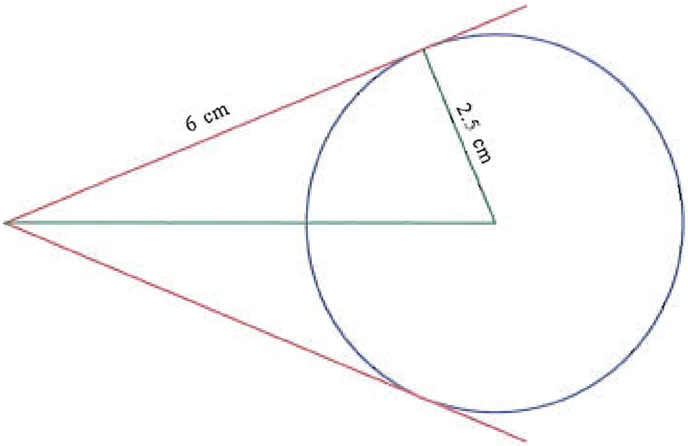
The method used in this work is to start from some basic assumptions, prove some simple facts using these, and go on to prove more and more complex theorems using these. (An online version of this work is available at http: // aleph0.clarku.edu/~djoyce/java/elements/elements.html)

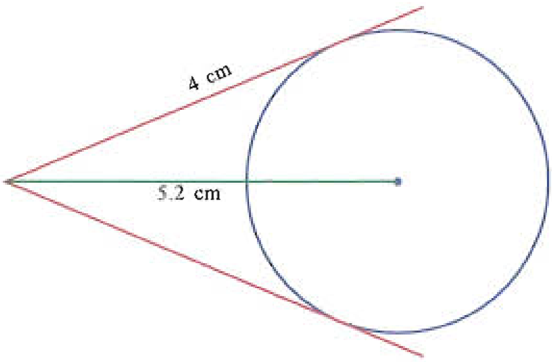
This method is now used not only in geometry, but in all branches of mathematics. Even in other sciences, we can see this method being used more or less.

Whatever be the way we discover mathematical theorems, the current practice is writing proofs the Euclidean way, giving each conclusion concisely, each a logical consequence of the earlier one.

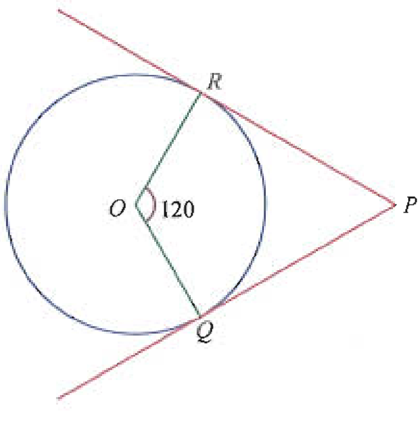
- Draw the figures below according to the specifications given.





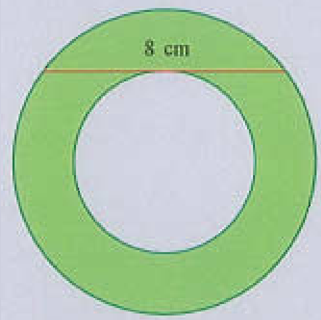


- In the picture below, the radius of the circle is 15 centimetres. Compute the lengths of the tangents *PQ* and *PR*



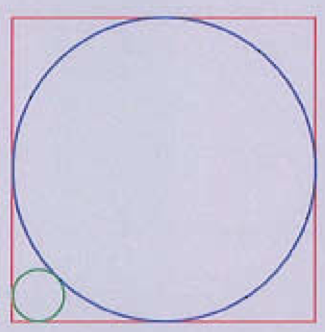
Area problem

What is the area of the green region in the figure below:



Corner problem

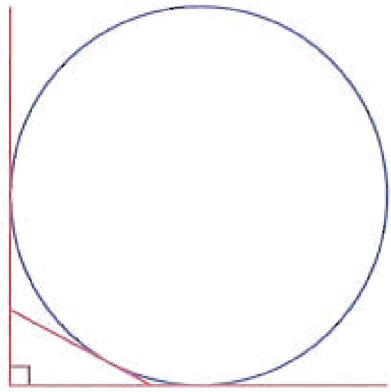
In the figure below, the large circle touches all four sides of the square and the small circle touches two sides of the square and the large circle:



What is the ratio of the radii of the two circles?

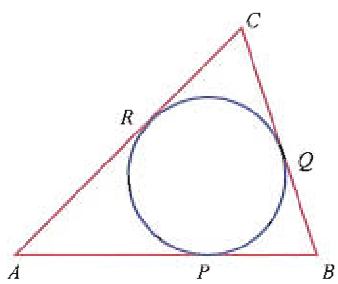
- In the circle centred at O, the tangents at A and B intersect at P. Prove the following:
  - the point P is equidistant from A and B
  - the line OP bisects the line AB and the angle APB
  - if the line OP cuts the line AB at Q, then $OQ \times OP = r^2$, where r is the radius of the circle

- The circle in the figure below touches all the three lines.



Prove that the perimeter of the right angled triangle is equal to the diameter of the circle.

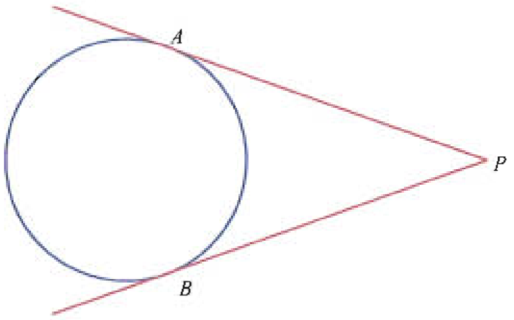
- In the figure below, the lines AB, BC, CA touch the circle at P, Q, R.



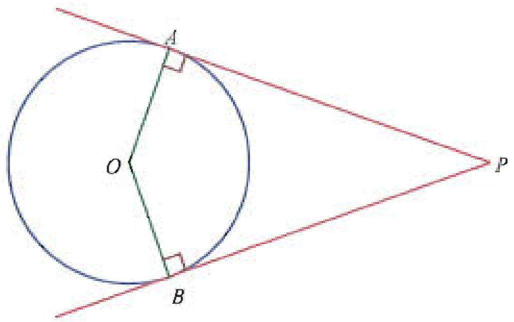
Prove that the perimeter of the triangle is $2(AP + BQ + CR)$.

Tangents and angles

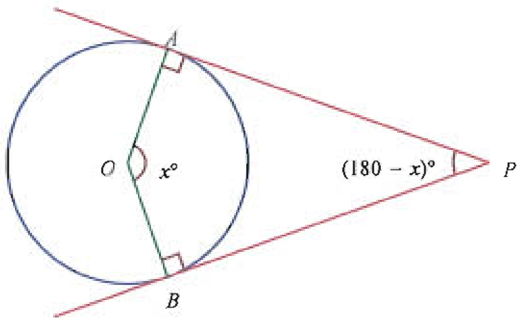
Suppose that the tangents to a circle at two points intersect at a point.



Look at the central angle of the small arc from A to B and the angle between the tangents at P.

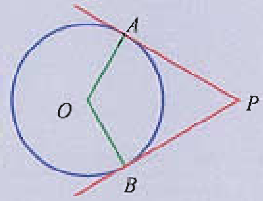


Two angles of the quadrilateral $OAPB$ are right and so their sum is 180°; this means the sum of the other two angles is also 180°. (What is the sum of all four angles of a quadrilateral?)

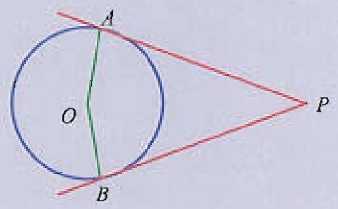


Near and afar

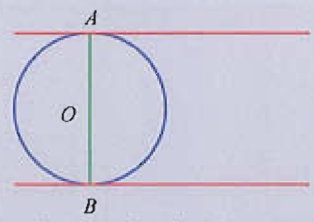
See this figure:



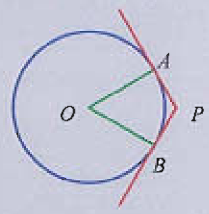
As $\angle AOB$ becomes larger, $\angle APB$ becomes smaller; moreover, P moves further away from O:



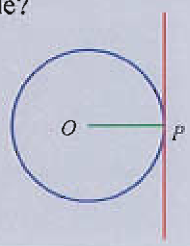
Finally what happens when AB becomes a diameter?



On the other hand, what happens as $\angle AOB$ becomes smaller?

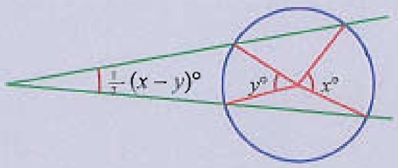


And finally, what happen when A and B coincide?

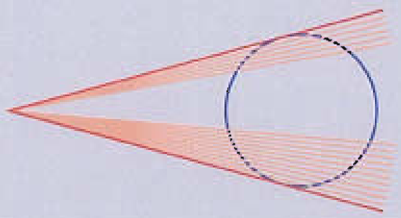


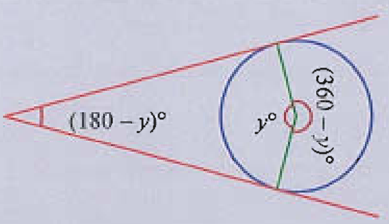
Chord and tangent

We have discussed the angle between two chords intersecting outside a circle in the section **Outside a circle**, of the lesson **Circles**:



What happens if these chords rotate about O and become tangents?

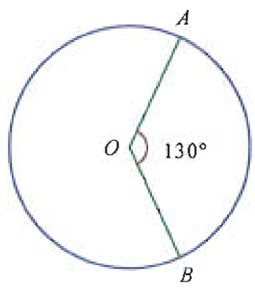


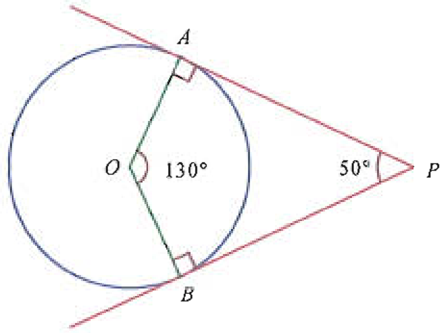


What do we see here?

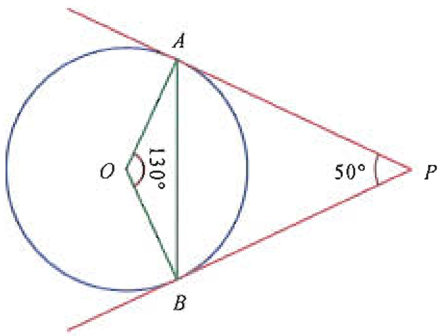
The central angle of the smaller arc between two points on a circle and the angle between the tangents at these points are supplementary.

For example, if we are asked to draw two tangents to a circle with the angle between them equal to 50°, we need only draw tangents at the ends of an arc of central angle 130°.





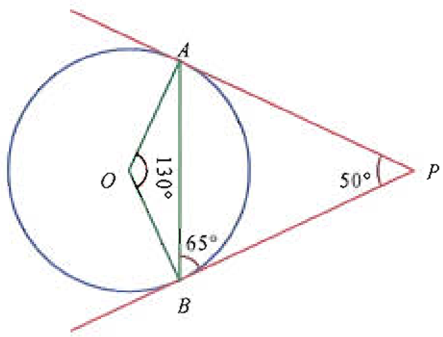
Suppose in this picture, we draw the chord AB also.



What is the angle between this chord and the tangents?

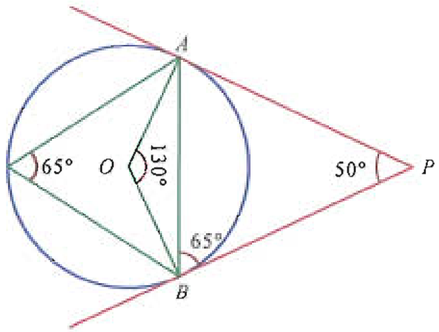
In the isosceles triangle OAB, the smaller angles are 25° each.

So, $\angle ABP = 90° - 25° = 65°$.



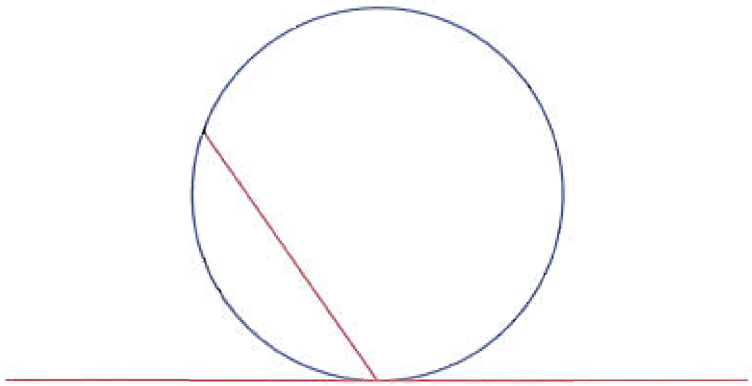
That is, half of 130°. But this is the angle in the larger segment cut off by the chord AB.

See this picture:

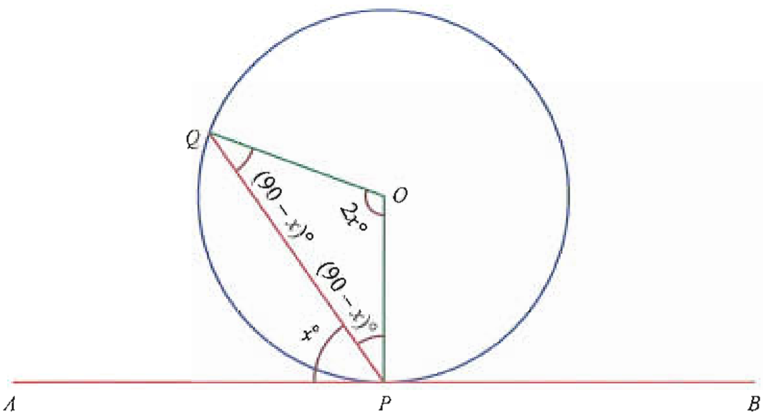


Is this true for all tangents and chords?

Let's draw a chord and the tangent at one of its ends.



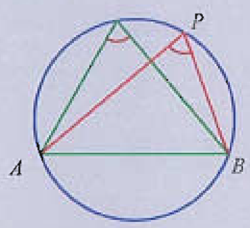
If we take one angle between the chord and the tangent as $x°$, then we can see from the figure below that the central angle of the smaller arc is $2x$. (Can you explain?)



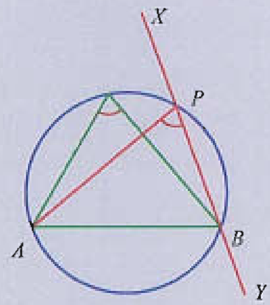
So, the angle made by the chord PQ in the larger segment is also $x°$.

Unchanging angle

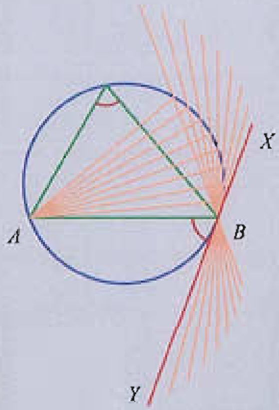
We have seen that angles in the same segment are equal:



Let's extend the line PB.



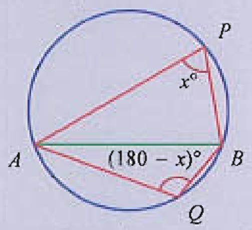
Now what happens as P moves along the circle to B?



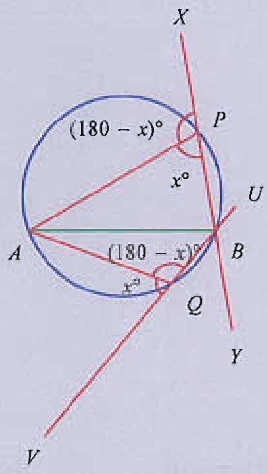
The line XY becomes the tangent at B; and the angle doesn't change.

Flip-flop angles

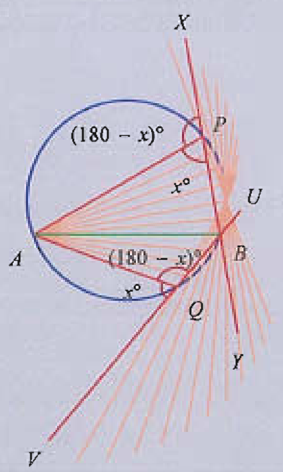
We have seen that angles in opposite segments of a circle are supplementary:



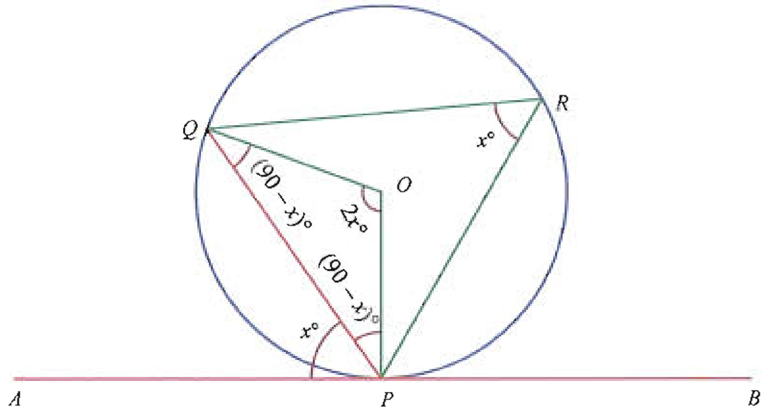
Let's extend the lines as before:



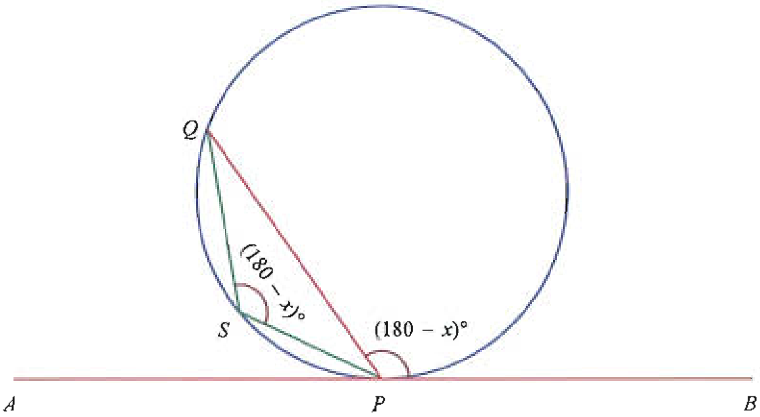
Suppose P moves along the circle to Q:



The angle below AP is $x°$ and the angle above AP is $(180 - x)°$. And this is so throughout the motion.



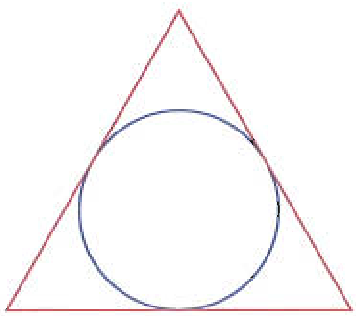
Not only this, but we can also see that the larger angle between the chord and the tangent and the angle in the smaller segment cut off by the chord are both equal to $(180 - x)°$.



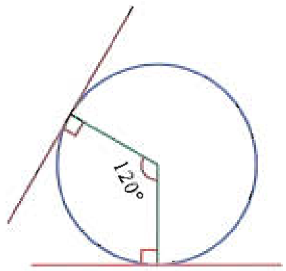
Let's write what we have seen now also as a general principle:

Each angle between a chord and the tangent at one of its ends in a circle is equal to the angle in the segment on the other side of the chord.

Using what we have seen about the angle between the tangents, we can solve our first problem about the equilateral triangle covering a circle.



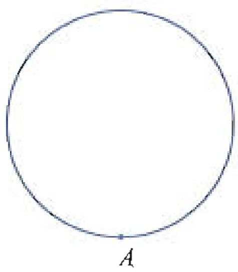
Here, the sides of the triangle are tangents to the circle, right? What is the angle between two of them? So, then what is the central angle of the arc between them?



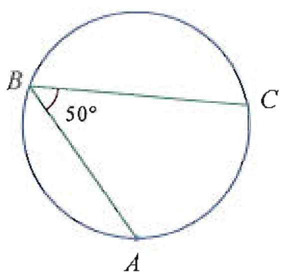
Now can't we draw the triangle? Can we draw nonequilateral triangles touching the circle like this? Try!

In this, we used the centre of the circle to draw the triangle. Can we do it without using the centre? (Suppose that we have a circle in which the centre is not marked.)

First let's see how we can draw a tangent without using the centre.

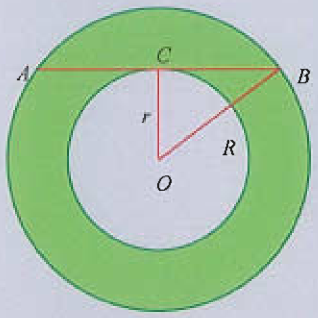


In the circle shown above, we want to draw the tangent at the point A. First draw two lines as shown below.



Now join AC and draw the line PQ through A, making an angle of 50° with AC.

Area problem—solution

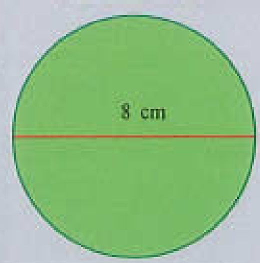


If we take the radius of the small circle as r and the radius of the large circle as R, then the area of the green region is $\pi(R^2 - r^2)$, right?

In the picture, AB is a tangent to the small circle and so it is perpendicular to the radius OC. Since AB is also a chord of the large circle, we also get $AC = BC$ (how?) So, from the right angled triangle OCB, we get $R^2 - r^2 = 4^2 = 16$.

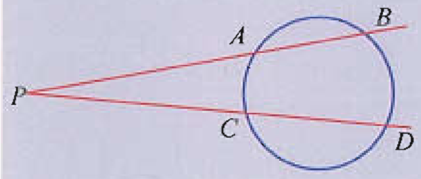
Thus the area of the green region is seen to be 16π square centimetres.

That this area is also equal to the area of the circle with diameter AB, is another point of interest:



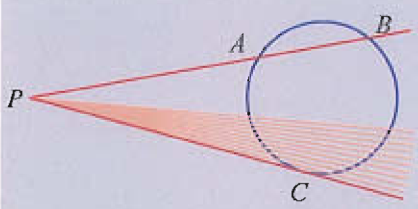
Unchanging relation

See this picture:



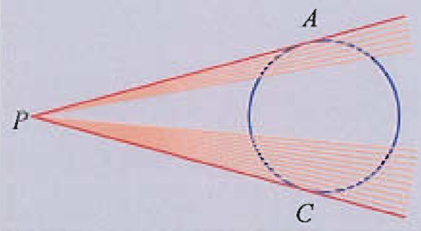
We know that $AP \times PB = CP \times PD$.

What if the bottom line turns to become the tangent at C?



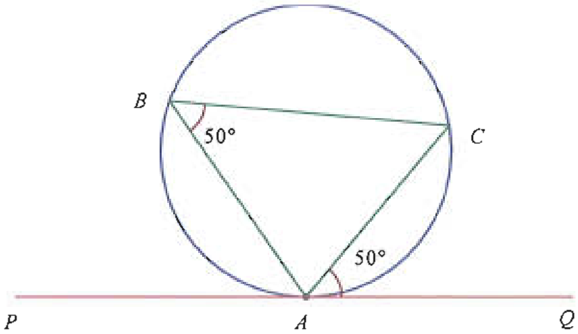
PD and PC become the same; and the relation between lengths becomes $AP \times PB = CP^2$

Suppose now the top line also turns to become the tangent at A:



The relation becomes, $PA^2 = PC^2$; that is $PA = PC$.

We have already seen that the lengths of the tangents from a point are equal, haven't we?

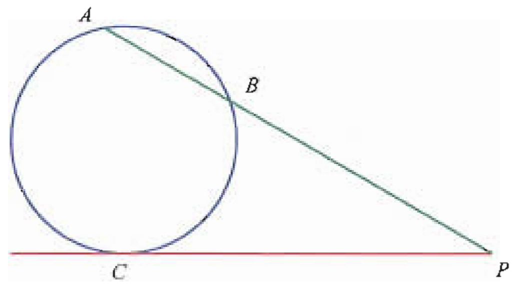


This is the tangent at A, isn't it?

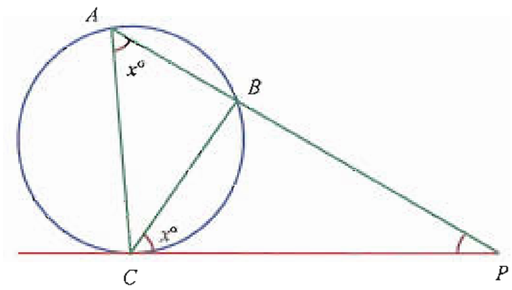
Can't we take any angle instead of 50° here?

Using the theorem about the angle between chord and tangent, we can form another general principle:

See this picture:



Join AC and BC. Taking $\angle BCP = x°$, we also get $\angle BAC = x°$.



That is, the angle at A in ΔAPC and the angle at C of ΔBPC are equal. Also, the angle at P is the same for both triangles. So, their third angles must also be equal. This means the pairs of sides opposite equal angles must have the same ratio. By a proper choice of such pairs, we get

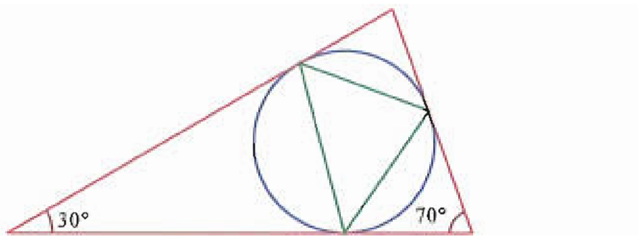
$$\frac{PA}{PC} = \frac{PC}{PB}$$

We can rewrite this as

$$PA \times PB = PC^2$$

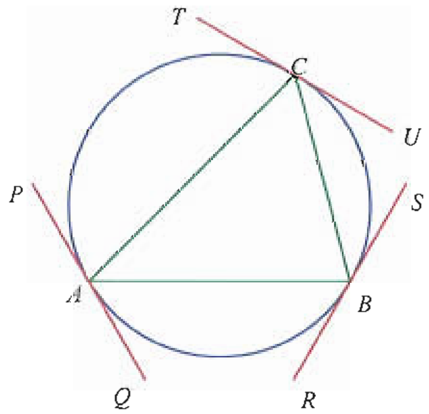
Now try these problems on your own:

- Draw a circle of radius 3 centimetres and draw a rhombus with one angle 40°, all four sides touching the circle.
- Draw a circle of radius 4 centimetres and draw a regular pentagon with all its sides touching the circle.
- Prove that in any circle, the tangents at two points make equal angles with the chord joining the points of contact.
- In the figure below, all the vertices of the small triangle are on the circle and all the sides of the larger triangle touch the circle at these points.



Find all angles of the small triangle.

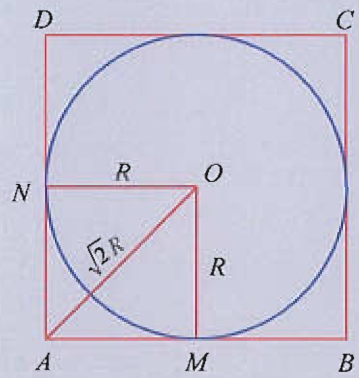
- In the picture below, PQ, RS, TU are tangents to the circle at A, B, C.



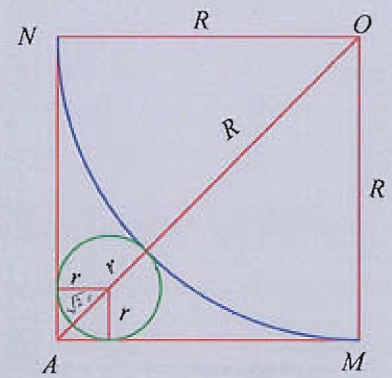
How many pairs of equal angles are there in it?

Corner problem—solution

Let's take the radius of the large circle as R. Drawing perpendiculars from its centre to the sides of the square gives the figure below:



We can do this with the small circle also. Let's take its radius as r. To see things clearly, we show below an enlarged view of a portion of the figure:



If we compute the length of OA from the two pictures, we get

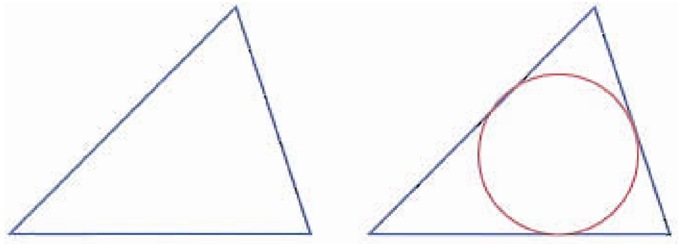
$$\sqrt{2}R = \sqrt{2}r + r + R$$

From this we get

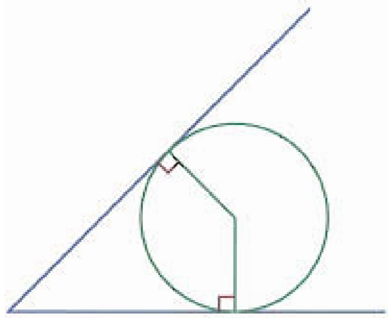
$$\frac{r}{R} = \frac{\sqrt{2}-1}{\sqrt{2}+1}$$

Inner circle

We have seen how we can draw a triangle with its sides touching a circle. Now let's see how we can draw a circle touching the sides of a triangle.

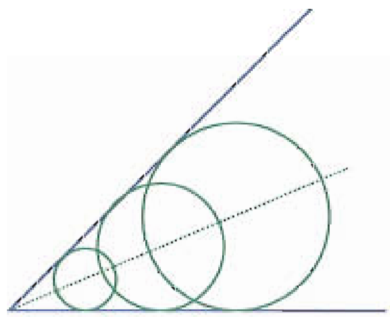


The circle should touch all three sides of the triangle. We can draw several circles touching a single side, what about circles touching a pair of sides?



In the picture, the radii are perpendicular to the pair of sides . In other words, the centre of the circle must be equidistant from these sides. So, it must be on the bisector of the angle between these sides (the section, **Equidistant bisector** of the lesson **Congruent triangles** in the Class 8 textbook)

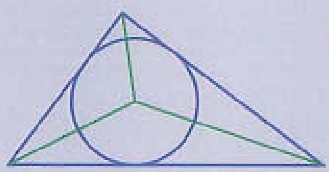
We can draw a circle touching the two sides, centred at any point on the angle bisector.



The circle we seek must touch the third side also. What do we do to get it?

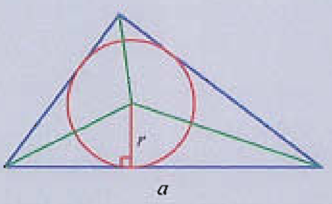
Radius of the incircle

We can compute the radius of the incircle of a triangle from the lengths of its sides. See this picture:

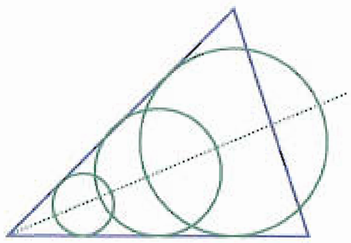


By joining the incentre with the three vertices of the triangle, we can divide the triangle into three smaller ones. The area of our triangle is the sum of the areas of these small triangles.

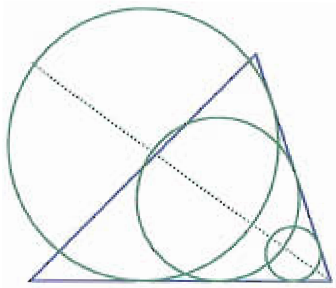
Now look at this picture:



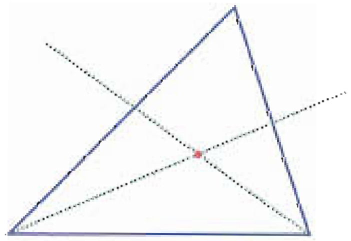
If we denote the inradius by r and the length of the bottom side of the triangle by a, the area of the small triangle at the bottom is $\frac{1}{2}ar$. Similarly, if the lengths of the other two sides are denoted b and c, the areas of the other two small triangles can be found as $\frac{1}{2}br$ and $\frac{1}{2}cr$. So, the area of our original triangle is $\frac{1}{2}(a + b + c)r$. So, the area of the triangle divided by half the perimeter gives the inradius.



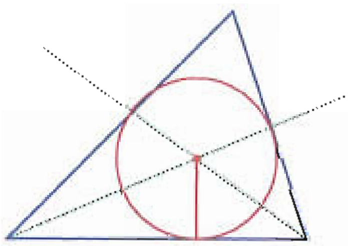
By taking points on the bisector of the angle between the bottom and right sides of the triangle as centres, we can draw circles touching these two sides.



So, how about taking the point on both these two angle bisectors? That is, the point of intersection of these bisectors.



From this point, the distances to all three sides are equal, right? So, what about the circle with this length as radius, centred on this point?

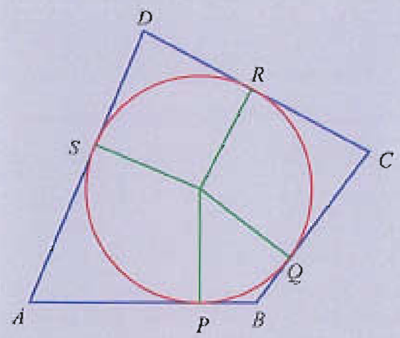


This circle is called the *incircle* of the triangle.

We can note another fact here. Since the incircle touches the left and right sides of the triangle, the perpendicular distances from these sides to the centre of the circle are equal; this means the incentre is the bisector of the angle between these sides also.

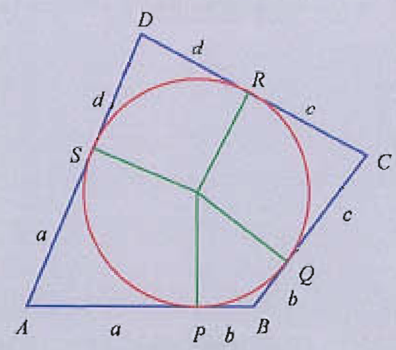
Quadrilateral and circle

See this picture:



The quadrilateral $ABCD$ has an incircle. P, Q, R, S are the feet of the perpendiculars from its centre to the sides of the quadrilateral.

Using the fact that tangents intersect at a point equidistant from the points of contact, we can mark lengths as below:



From this, we can see that

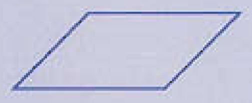
$$AB + CD = a + b + c + d = AD + BC$$

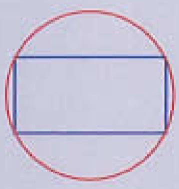
That is, if a quadrilateral has an incircle, then the sums of its opposite sides are equal.

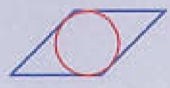
On the other hand, we can prove that any quadrilateral with the sums of the opposite sides equal, has an incircle (Try it!)

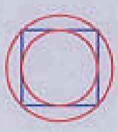
Circumcircle and incircle

We can draw a circumcircle and an incircle for any triangle. But when we come to quadrilaterals, some may have neither, some may have one and not the other, and some may have both:

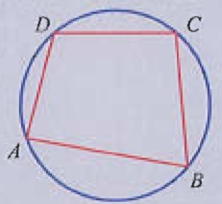






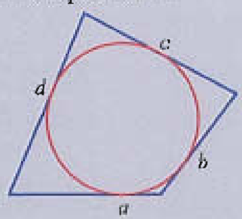


We have seen that in a quadrilateral which has a circumcircle, the sum of the opposite angles is 180°. In other words, both pairs of opposite angles have the same sum:



$$\angle A + \angle C = \angle B + \angle D$$

What about quadrilaterals which have incircles? Both pairs of opposite sides have equal sum:



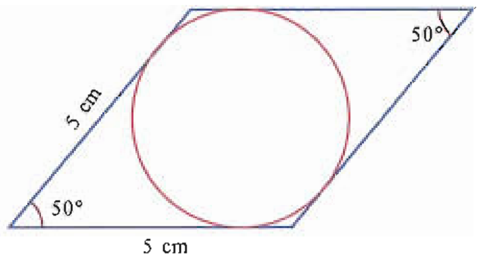
$$a + c = b + d$$

And this is true for any triangle, isn't it?

In any triangle, the angle bisectors meet at a point.

Now some problems for you:

- Draw a triangle of sides 4, 5, 6 centimetres and draw its incircle.
- Draw an equilateral triangle of sides 6 centimetres and draw its incircle and circumcircle.
- Prove that in an equilateral triangle, the circumcentre and incentre are the same. What is the ratio of the circumradius and inradius?
- Draw a square of sides 5 centimetres and draw its circumcircle and incircle.
- Draw the figure below according to the given specifications:



Project

- Find different methods of drawing line segments of lengths $\sqrt{2}, \sqrt{3}, \sqrt{5}$ using the following ideas:
  - Pythagoras Theorem
  - Theorems on chords of circles
  - Theorems on tangents to circles