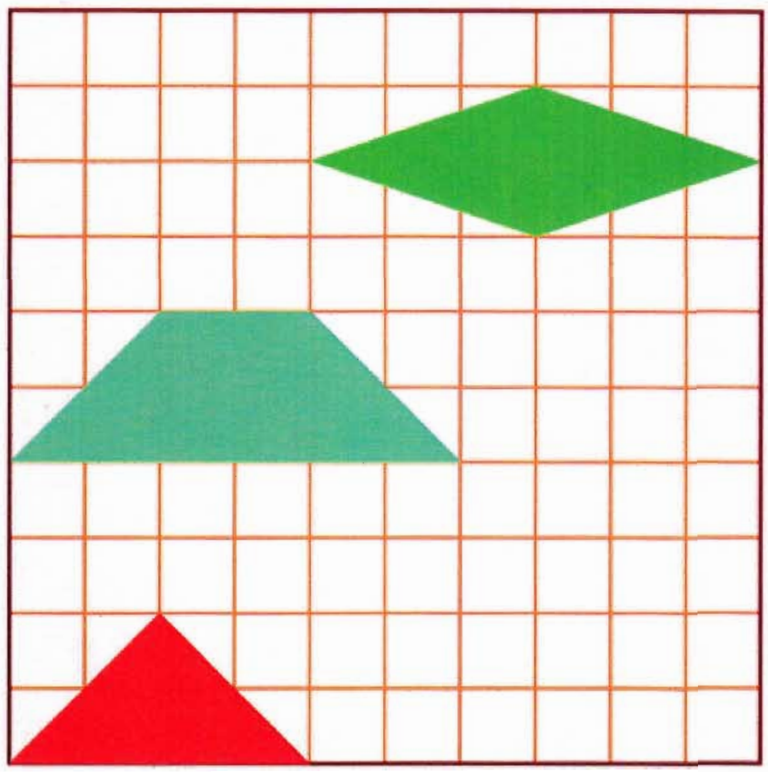
6 Coordinates

Pictures and numbers

See the picture below.



We have drawn such pictures in Class 9 (see the section **Drawing Polygons**, of the lesson **Polygons**).

How do you make a copy of this picture?

First draw a 10 centimetre square; then draw lines across and down, 1 centimetre apart, and thus divide it into small squares. What next?

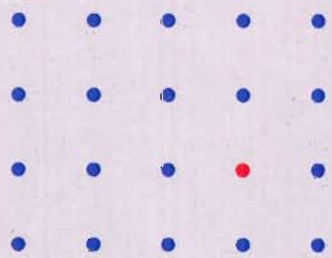
Let's first draw the rhombus at the top. For this, we must mark its four vertices. Where do we put the left vertex?

It is at a point where a line across and a line down meet. Which lines?

Rows and columns

From things arranged in rows and columns, how do we indicate a thing at a specific position? For example, from the books in a shelf, we can say that the one we want is, "the one in the third row from the bottom, fifth from the left", or something similar.

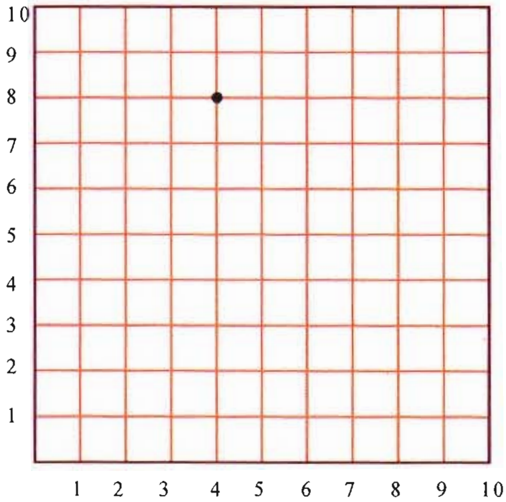
See this picture:



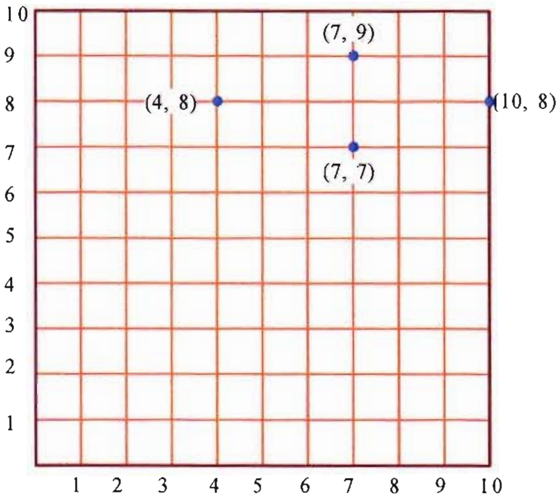
How do we indicate the position of the red dot in this?

We can say, "the dot in the second row from the bottom, fourth from the left". Any other ways of specifying this position?

The line 4 centimetres to the right from the left of the large square and the line 8 centimetres up from its bottom.



Similarly, we can specify the positions of the other vertices, in terms of the distances from the left and the bottom of the large square. For convenience of reference, let's write these numbers alongside the points.



Now we can draw the rhombus. Moreover, we can easily tell the others where to draw it.

Likewise, how do we specify the vertices of the trapezium in the first picture?

Its bottom-left corner is on the left line itself. Let's indicate this line and the bottom line of the square as 0 (Why?)

Position in a table

A table consists of cells arranged in rows and columns. How do we specify a certain cell?

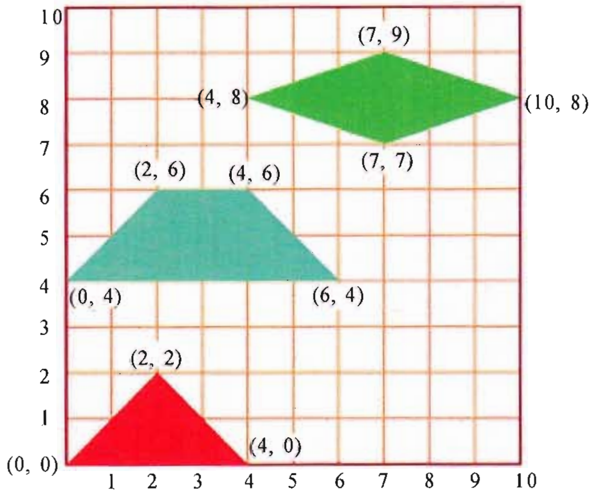
Aren't you familiar with spreadsheet programs such as Open Office Calc? How do you specify cells in a spreadsheet?

The rows of the table are indicated by the numbers 1,2,3 ,.. on the left, running from the top to the bottom; and the columns are indicated by the letters A, B, C, ... at the top, running left to right. Using these two, we can specify any cell.

For example, in the picture above, the number 100 is in the cell F6.

So, how do we write the bottom-left vertex of our trapezium? What about its other vertices?

What numbers give the vertices of the triangle in the picture?

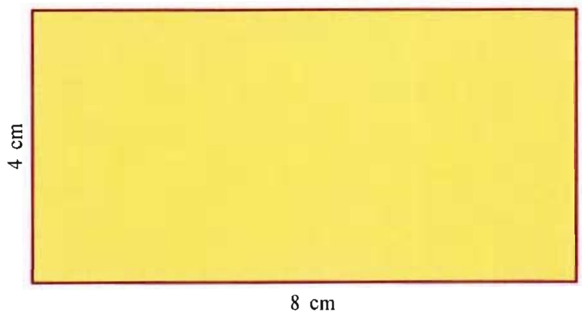


Now make a new grid like this and draw the figures listed below; also indicate the positions of their vertices using numbers as done above:

- a non-isosceles triangle
- a parallelogram which is not a rhombus
- a non-isosceles trapezium
- a pentagon
- a hexagon

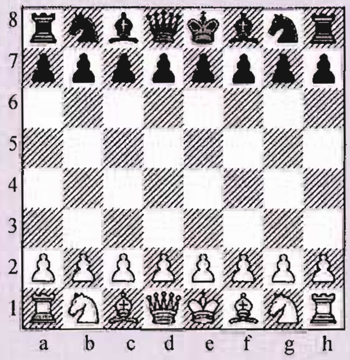
More on numbers and pictures

From the middle of a rectangle 8 centimetres wide and 4 centimetres high, we want to cut out a rectangle 4 centimetres wide and 2 centimetres high.



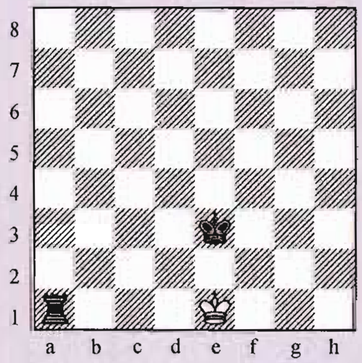
Chess positions

Have you noticed how the positions on the board are specified in descriptions of chess matches?

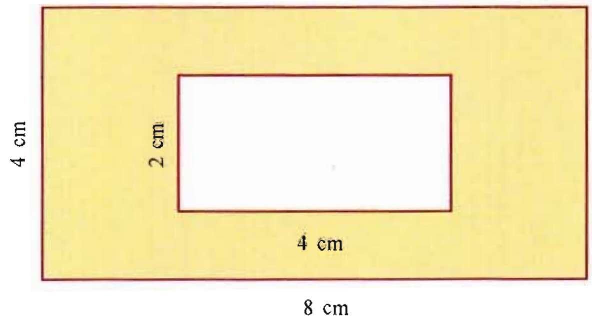


The rows are named with numbers and the columns with letters, as shown above.

Now see this picture:



The position of the White King is e1 and the position of the Black King is e3. The Black Rook is at a1.

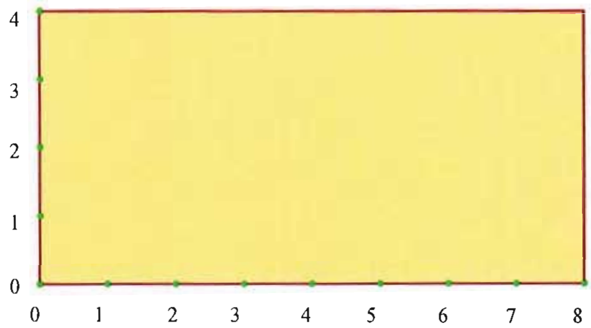


We first mark the vertices of the rectangle to be cut out in the large rectangle.

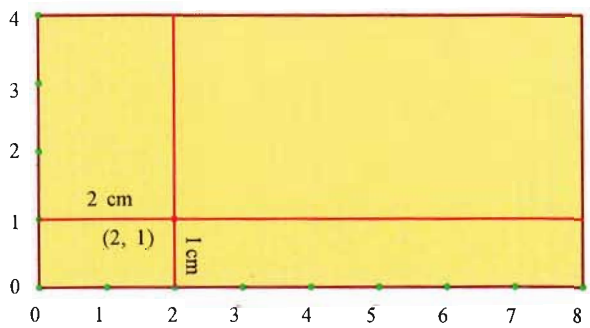
Since the small rectangle is to be right in the middle of the large rectangle, its left and right sides must be at the same distance from the left and right sides of the large rectangle; the same goes for the top and bottom sides.

At what distance?

Let's mark distances along the left and bottom lines of the large rectangles, one centimetre apart.



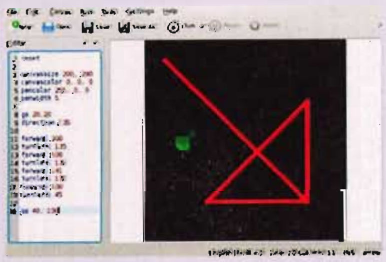
Now how do we mark the bottom-left corner of the rectangle to be cut off?



We can mark the other vertices also like this.

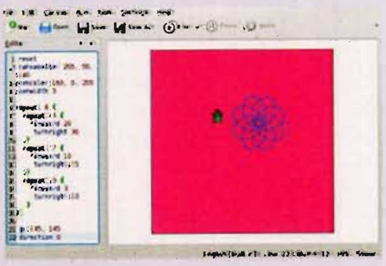
Computer pictures

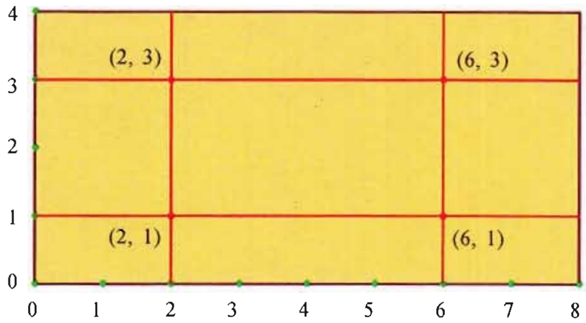
There's a simple program called KTurtle in Gnu/Linux which helps us to draw geometrical shapes with a computer. It is done by specifying positions on the screen in terms of numbers.



The left pane in the picture shows the code used to draw the picture in the right pane.

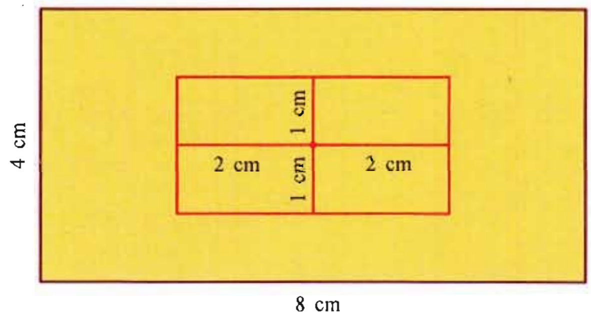
With a little effort, complex diagrams can also be drawn with it.



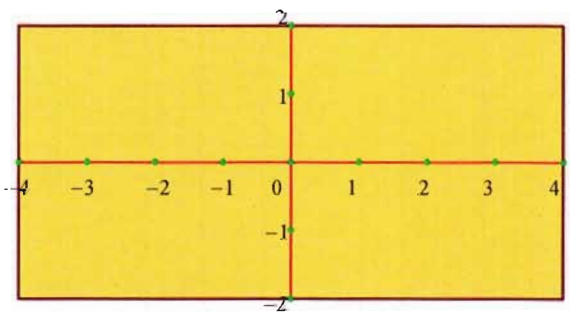


Now we can draw the rectangle and cut it out.

We can think about this in a slightly different way. The left and right sides of the rectangle to be cut out are 2 centimetres to the left and right of the centre of the large rectangle; its top and bottom are 1 centimetre up and down this point.



To specify the vertices of the small rectangle in this manner, let's draw a pair of lines across and down the large rectangle, through its centre. To distinguish between left-right and up-down directions, we write distances towards the left and distances down as negative numbers. (Remember how we marked the positions of numbers on the number line?)



How do we specify with respect to these lines, the vertices of the rectangle to be cut out?

Typesetting

Computers are now extensively used in typesetting–that is, the design of pages to be printed, specifying where in the page each letter or picture should be placed.

A language used for this purpose is PostScript. It uses numbers to specify the various locations in a printed page.

Let's look at an example. Use some text editor such as gedit in Gnu/Linux and type the lines below:

```
newpath
20 20 moveto
40 20 lineto
40 40 lineto
20 40 lineto
closepath
fill
showpage
```

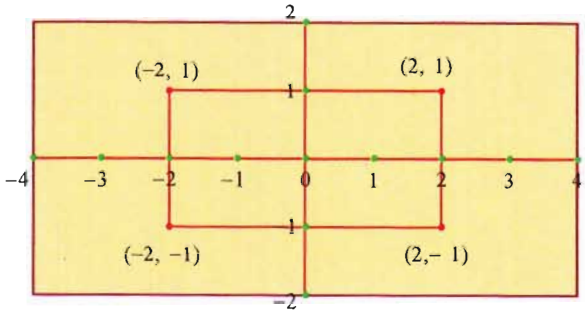
This is a sample of the PostScript language. To see what has been drawn, we can use a program such as gv. Save the file as **test.ps** and then run the command

gv test.ps

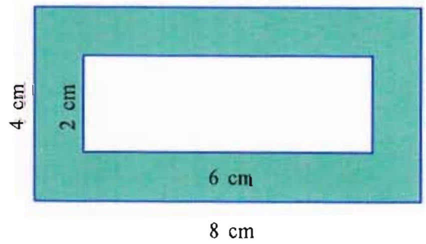
in a terminal. A small black square at the bottom left of a white screen will be displayed.

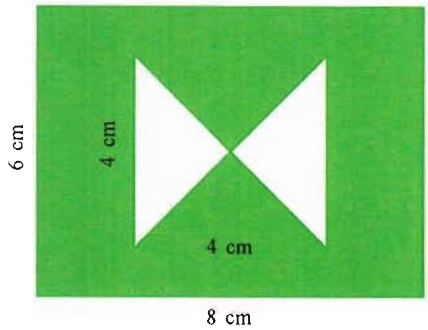
The number pairs used in this code are all distances from the left and bottom sides of the page. Distances are measured in point, which is the usual unit in printing. One point is about 0.035 centimetres.

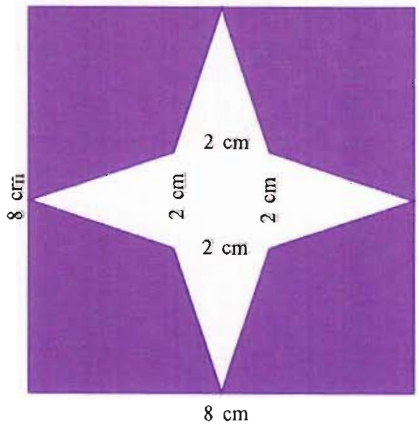
In most DTP applications, PostScript works invisibly in the background.



Now write the points to be marked for cutting out the figures shown below, using measurements from the centre of the rectangle, as done in the last example above.



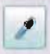




Colours and numbers

In a computer, not only positions on the screen, but the colours used are also specified by numbers. Various colours are produced by mixing different amounts of the basic colours red, green and blue.

This can be easily understood using the application Gcolor2 in GNU/ Linux.

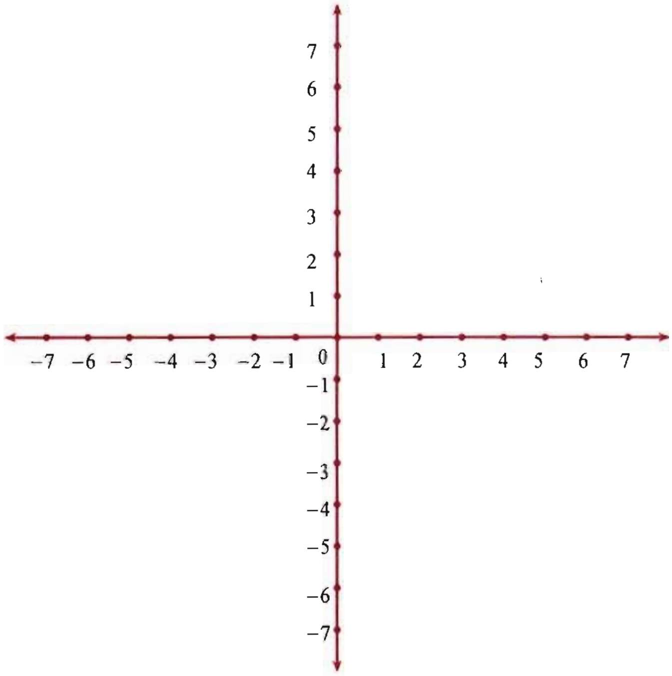
By first clicking on the tool  and then clicking on any part of the screen, we get the RGB values of the colour at that point.

Positions and numbers

We have seen some examples of how the positions of certain points in a plane can be specified by means of a pair of numbers. What does each pair of numbers represent?

The distances of the point from a pair of perpendicular lines, right?

See the figure below:



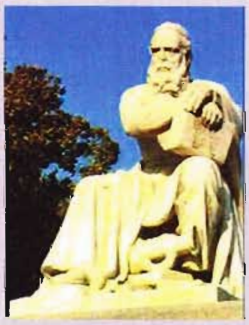
A pair of perpendicular lines, with distances from the point of intersection marked along each. Positive and negative numbers are used, to distinguish between right-left and up-down directions. The markings need not be 1 centimetre apart; the only requirement is that the distance between adjacent points should be the same throughout. In other words, we can use any unit for measuring distance. (We can think of these as two number lines drawn perpendicular to each other and sharing a common zero.)

Using the distances from these lines, we can specify any point in the plane.

A bit of history

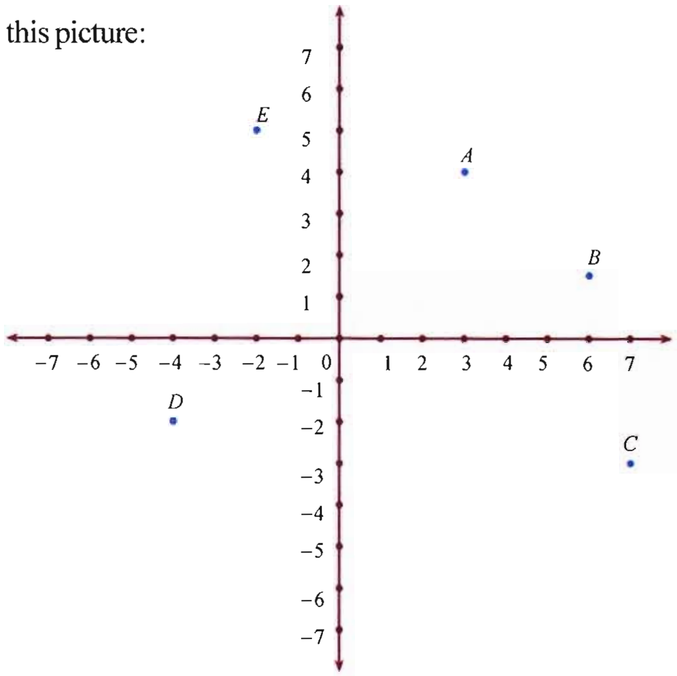
The technique of specifying the positions of points using numbers was used as early as the second century BC by the Greek mathematician Appollonius, in his solution of certain geometric problems. These numbers were the distances of the points from some fixed lines.

In the eleventh century AD, the Persian mathematician and poet, Omar Khayyam used the method of representing number pairs as points to convert certain algebraic problems to geometry.



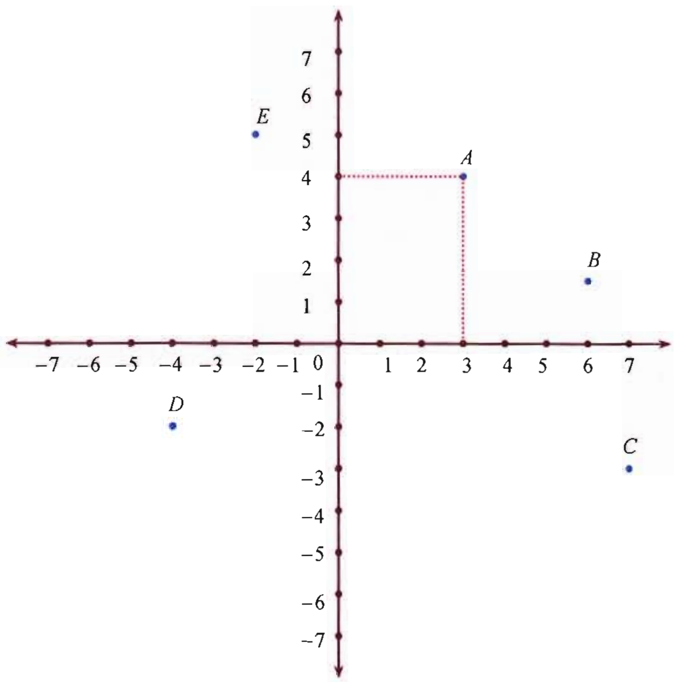
This relation between geometry and algebra developed as a definite branch of mathematics after the French mathematician and philosopher, Ren´e Descartes published his "Geometry", in the seventeenth century.

See this picture:



What are the number pairs of the points *A*, *B*, *C*, *D*, *E*?

Take *A*. Draw perpendiculars from *A* to the reference lines.



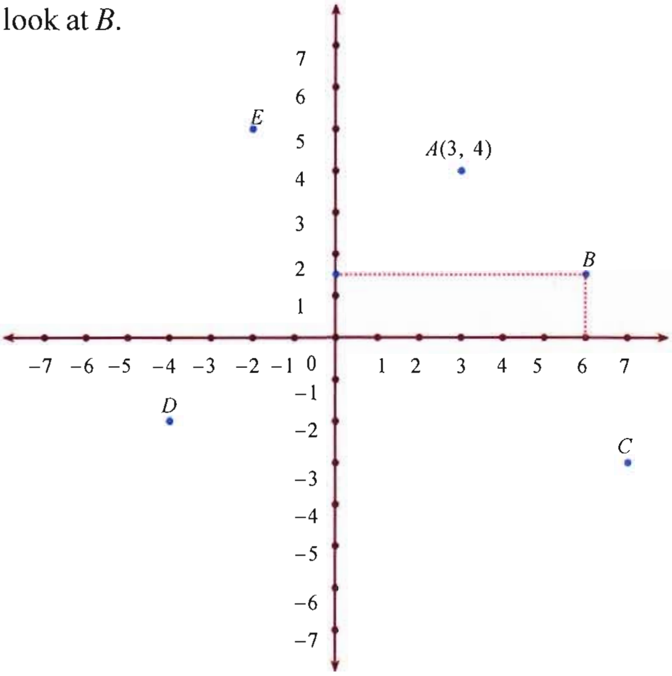
The perpendicular to the horizontal line meets it at the point marked 3, and the perpendicular to the vertical line meets at the point 4. So, as before, we can denote the point *A* by the pair (3, 4) of numbers.

Geometry and a fly

There's an intersesting story told about Descartes discovery of his new method in geometry. Lying on the bed and thinking, he notices a fly crawling on the ceiling. He is hit by the idea that to trace its path, he need only know how its distances from two adjacent walls change. And this leads him to a method of reasoning in geometry.

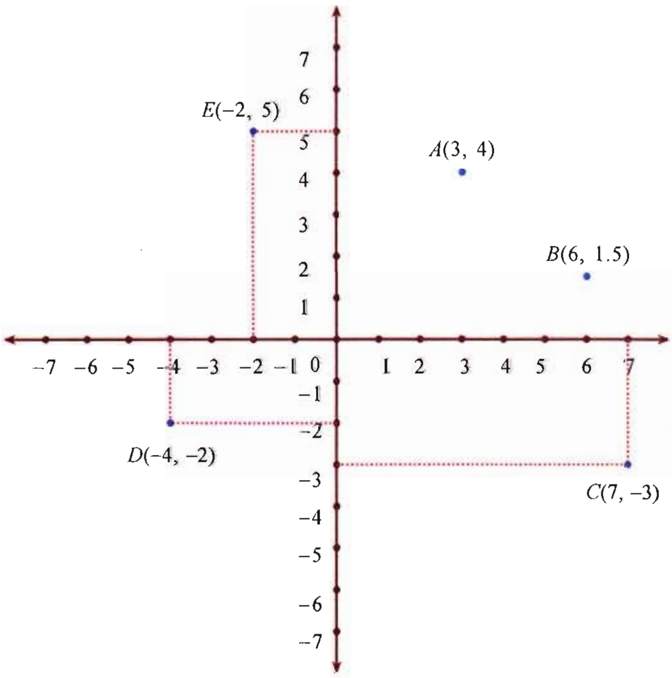
Whether this story is true or not, it is a nice illustration of the fact that using the distances from two reference lines, we can indicate the position of any point in the plane and that geometric figures can be described using such pairs of numbers.

Now look at *B*.



The perpendicular to the vertical line meets it exactly at the middle of 1 and 2. So, we can write *B* as $(6, \frac{3}{2})$

In the same way, we can also denote the remaining points *C*, *D* and *E* by number pairs.



Thus, using a pair of line perpendicular to each other and a suitable unit for measuring length, we can denote every point in a plane by means of pairs of numbers.

Dividing the Earth

We use latitudes and longitudes to locate places on the earth. What is their meaning?

First let's see how we place a grid over the earth, as we did on squares and rectangles.

The earth rotates; and in any rotating sphere, there are two points which don't move. These are the *poles* and the line joing them is the *axis of rotation*. By a *great circle* on a sphere, we mean a circle on the sphere with the same centre as that of the sphere. The great circle on the earth equidistant from the poles is *the equator*. Other circles on the earth parallel to the equator are the lines of *latitudes*. Great circles through the poles are the lines of *longitudes* (also called *meridians*). Of these, the one that passes through Greenwhich village in England is the *prime meridian*.

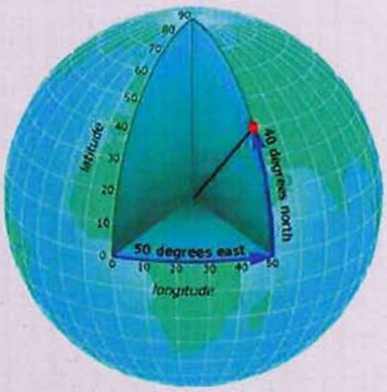
Thus we can imagine the grid of latitudes and longitudes covering the earth:

Locations on the Earth

We saw how the earth is divided into cells by the latitudes and the longitudes. We use this to indicate any position on the earth.

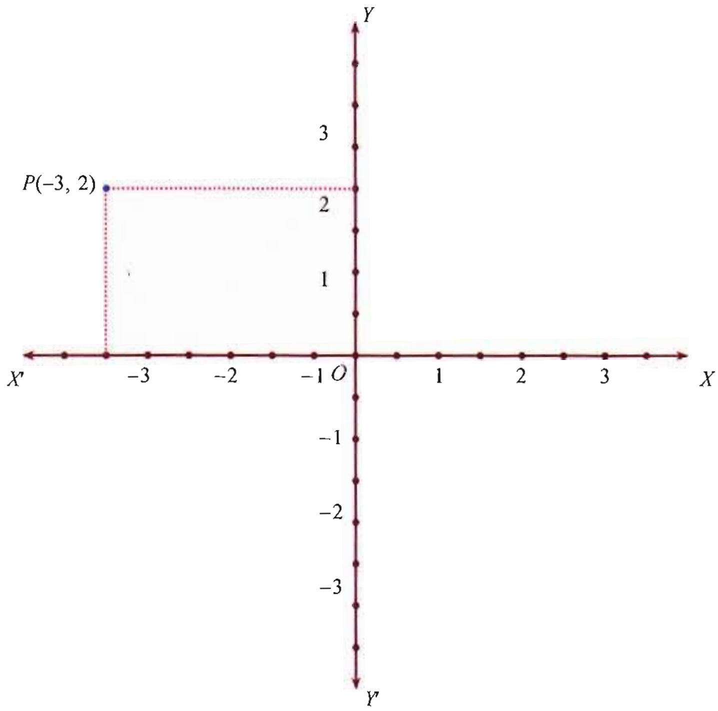
For this, we first suppose a line joining the centre of the earth with a point of intersection of the equator and the Greenwhich meridian. For this point to move to another latitude, it must move north or south; and so the line joining the point to the centre of the earth must move up or down through a certain angle. The latitudes are specified in terms of these angles (and also the adjectives north or south). On the other hand, what if our original point is to move to another longitude? It must move east or west; and the line should move left or right through a certain angle. These angles are the labels for the longitudes.

We can specify any position on earth using these angles.



The two reference lines are called the *axes of coordinates*; the horizontal line is called the *x-axis* and the vertical line is called the *y-axis*. The *x-axis* usually named XX' and the *y-axis*, YY'. Their point of intersection is called the origin and is usually named O.

We have seen the practice of drawing perpendiculars from a point to the coordinate axes, to indicate its position. The pair of numbers thus got are called the *coordinates* of the point; the first number is called the *x-coordinate* and the second number is called the *y-coordinate*.

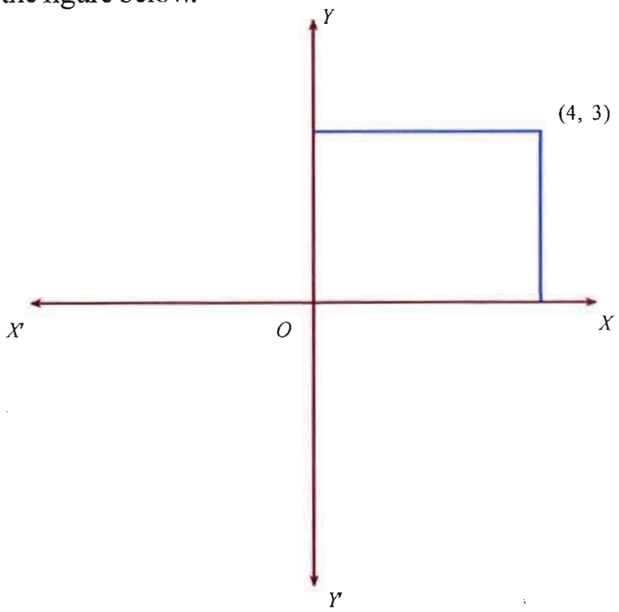


For example, in the figure above, the *x-coordinate* of the point P is −3 and the *y-coordinate* is 2.

Now some problems:

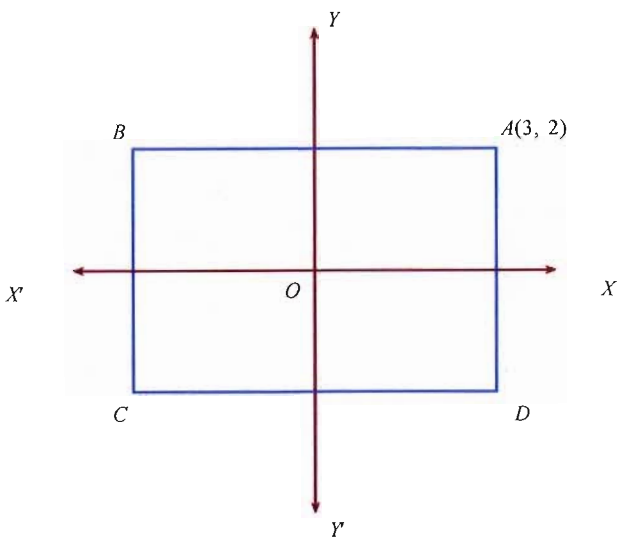
- What is the speciality of the y-coordinates of points on the *x-axis*?
- What is the speciality of the x-coordinates of points on the *y-axis*?
- What are the coordinates of the origin?

- Find the coordinates of the other three vertices of the rectangle in the figure below.



The unit of length used in this $\frac{3}{4}$ centimetres. What is actual width and height of this rectangle?

- In the figure below, *ABCD* is a rectangle with the origin *O* as its centre and sides parallel to the axes.



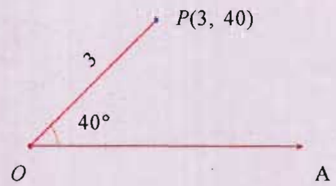
What are the coordinates of *B*, *C*, *D* ?

Distance and direction

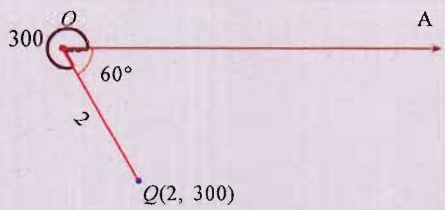
Instead of using the distances from two perpendicular lines to specify positions of points, we can use the distance from a fixed point and the angle made with a fixed line.

For this, take a point *O* and a line *A* through it:

Now for any point *P* in this plane, we can specify its position by means of the length *OP* and the angle *POA*.

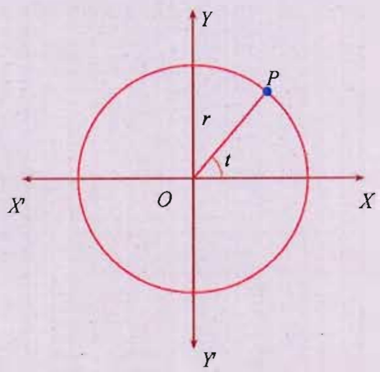


How do we specify the position of *Q* in this method?



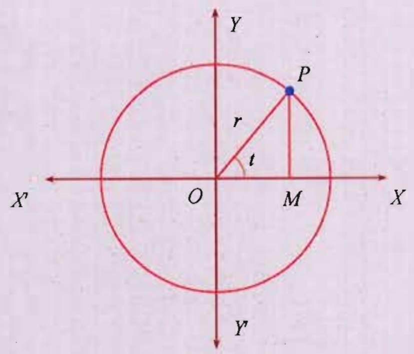
A little trigonometry

We draw a circle of radius r centred at the origin, and take a point P on it, as shown blelow:



Taking $\angle POX = t$, what are the coordinates of P?

Drawing the perpendicular PM from P to the x-axis, we get the right angled triangle POM.



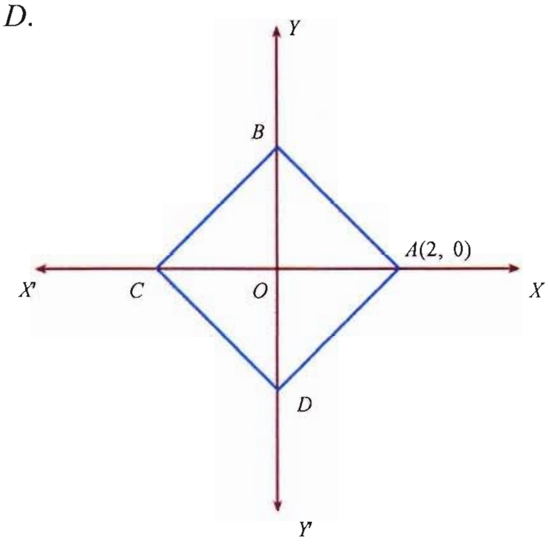
From the figure, we see that

$$OM = r \cos t. \quad PM = r \sin t$$

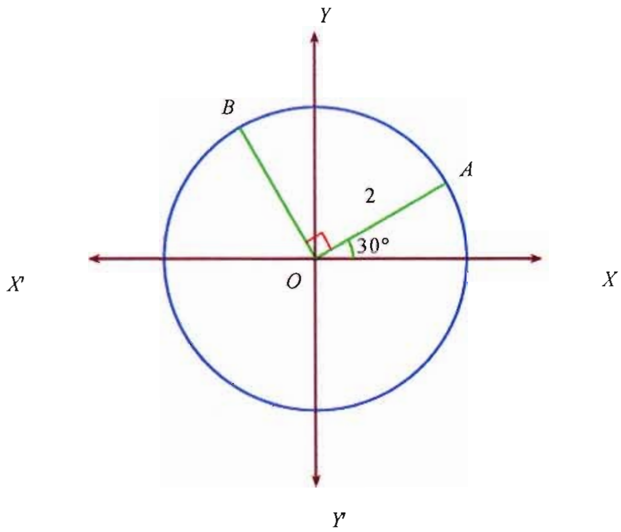
This means the coordinates of P are $(r \cos t, r \sin t)$.

What if $\angle POX$ is a right angle or larger?

- In the figure below, $ABCD$ is a square. Find the coordinates of B, C, D.



- What are the coordinates of the points A and B in the figure below?

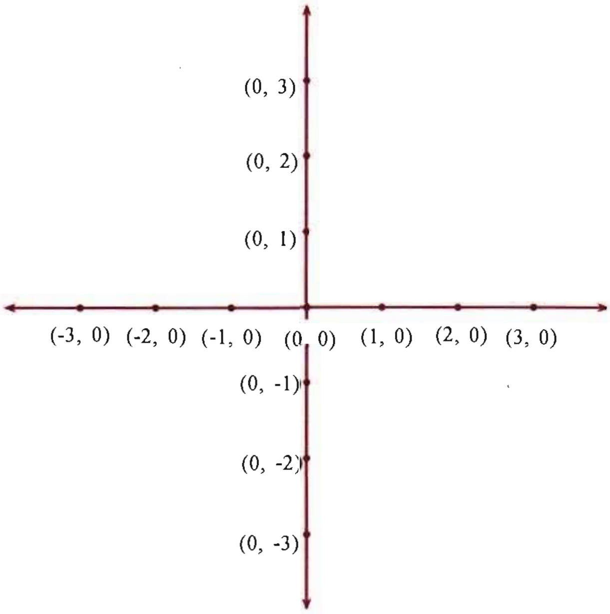


- With the axes of coordinates chosen along two adjacent sides of a rectangle, two opposite vertices have coordinates $(0, 0)$ and $(4, 3)$. What are the coordinates of the other two vertices?

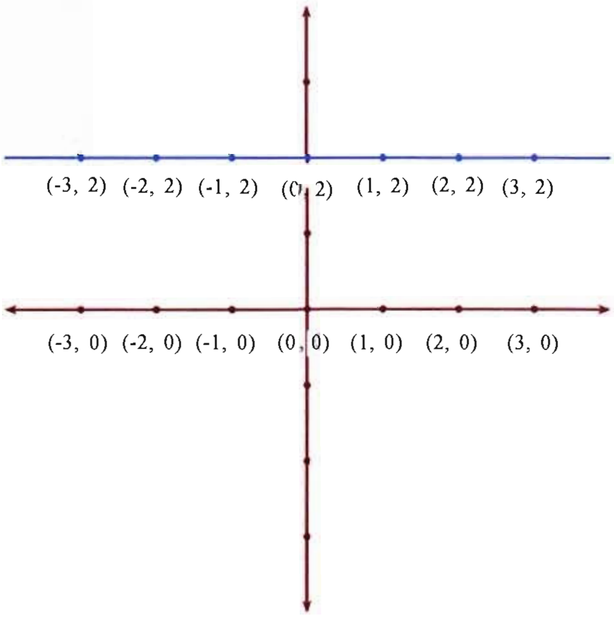
Parallels

We noted that all points on the x-axis have their y-coordinate 0; on the other hand all points with their y-coordinate 0 are on the x-axis. In other words, the x-axis can be described as the collection of all points whose coordinates are of the form $(x, 0)$.

Likewise, the y-axis is the collection of all points of the form $(0, y)$.

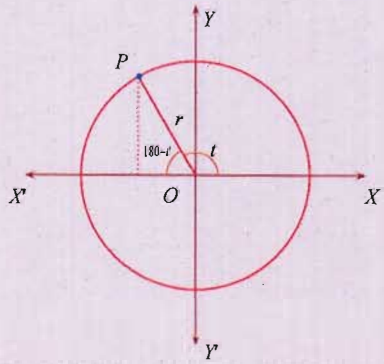


What if we take some other number instead of 0? For example, what about points of the form $(x, 2)$? All these are at a distance 2 (with respect to the unit of length chosen) from the x-axis. So, all these are on the line parallel to the x- axis, at a distance 2 from it.



Some more trigonometry

See this figure:



It can be directly seen from the figure itself that the x-coordinate of P is $r \cos(180 - t)$ and its y-coordinate is $r \sin(180 - t)$.

Also, t is an angle between 90 and 180, and we have defined

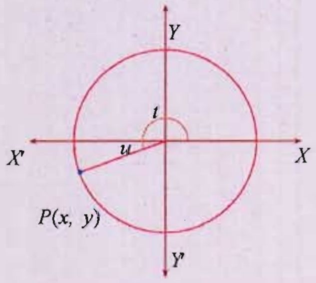
$$\cos(180 - t) = -\cos t$$

$$\sin(180 - t) = \sin t$$

in the lesson **Trigonometry**. So, here also, the coordinates of P are $(+\cos t, +\sin t)$

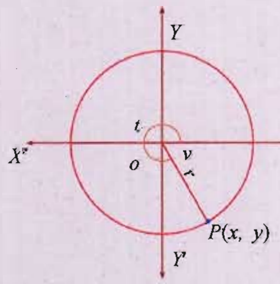
Trigonometry through circles

See this figure:



Here we can see that the coordinates of P are $(-r \cos u, -r \sin u)$; also $u = t - 180$. So, if we define, $\cos(180 + u) = -\cos u$ and $\sin(180 + u) = -\sin u$, then the coordinates of P would be $(r \cos(180 + u), r \sin(180 + u))$; and thus the coordinates of P would be $(r \cos t, r \sin t)$ in this case also.

What about the figure below?

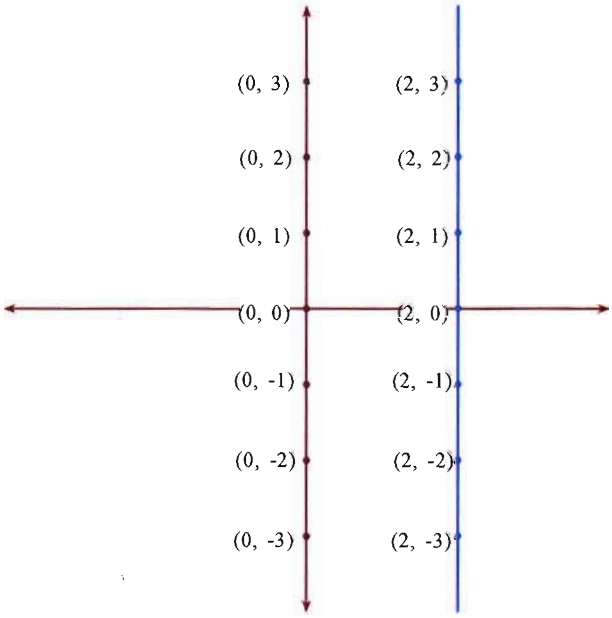


Here, the coordinates of P are $(r \cos v, -r \sin v)$ and $v = 360 - t$. So, if we define $\cos(360 - v) = \cos v$ and $\sin(360 - v) = -\sin v$, then the cordinates of P would be $(r \cos t, r \sin t)$ in this case also.

In short, we extend the definitions of $\sin t$ and $\cos t$ in such a way that the coordinates of P on the circle centred at O with radius r is $(r \cos t, r \sin t)$, where t is the central angle of the arc from the x-axis to P, however large t be.

And all points with y-coordinate 2 are on this line.

What about the collection of points of the form $(2, y)$?



In short, whatever number we take, the collection of all points of the form (x, a) is the line parallel to the x-axis, at a distance a from it; and the collection of all points of the form (a, y) is the line parallel to the y-axis, at a distance a from it.

In one sense, all these are number lines. We use labels like (x, a) or (a, y) to denote the points instead of single numbers like x or y.

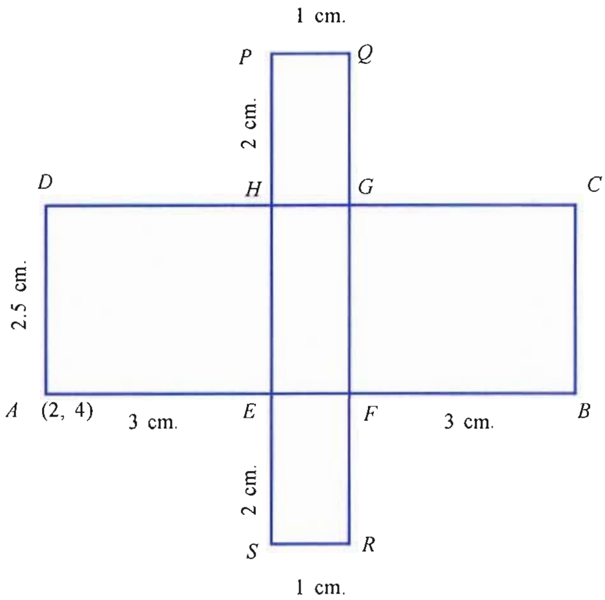
So then, a question: what's the distance between $(1, 2)$ and $(3, 2)$?

What about the distance between $(1, 2)$ and $(-3, 2)$?

As in a number line, we need only subtract the smaller x-coordinate from the larger, right?

In short, to find the distance between (x_1, a) and (x_2, a), we need only subtract the smaller of x_1, x_2, from the larger; in algebraic language, the distance is $|x_1 - x_2|$.

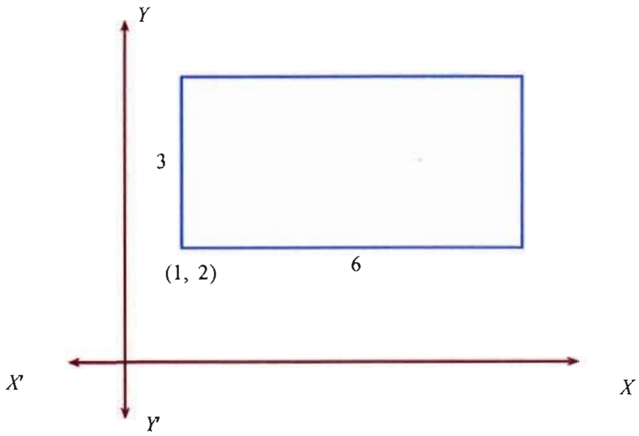
Similarly, to find the distance between (a, y_1) and (a, y_2), we need only subtract the smaller of y_1, y_2, from the larger; in algebraic language, the distance is $|y_1 - y_2|$.



In the figure, the sides of the rectangle *ABCD* and *PQRS* are parallel to the axes. Find the coordinates of the vertices of all the rectangles in it.

Rectangles

Look at this figure:

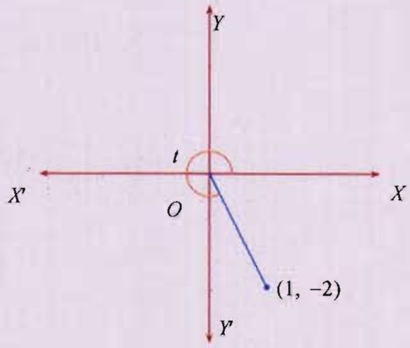


We want to draw a rectangle like this, with sides parallel to the axes. What should be the coordinates of the other three vertices?

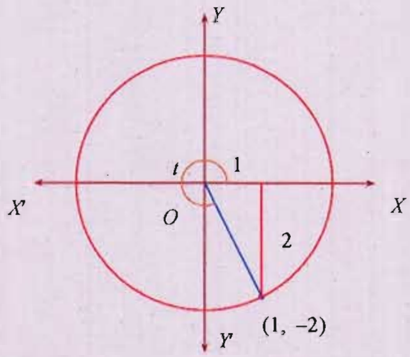
All points on the bottom side of the rectangle would be of the form $(x, 2)$ (why?) Among these, the bottom-right corner of the rectangle is 6 units away from $(1, 2)$. So, what are its coordinates?

Without circles

What is the sin and cos of the angle t in the figure below?



We can imagine a circle through $(1, -2)$, centred at the origin:



We can see from the figure that the radius of the circle must be

$$\sqrt{1^2 + 2^2} = \sqrt{5}$$

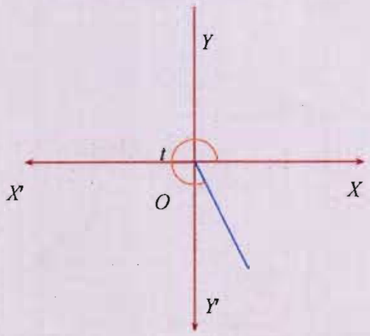
So, by definition

$$\cos t = \frac{1}{\sqrt{5}} \quad \sin t = -\frac{2}{\sqrt{5}}$$

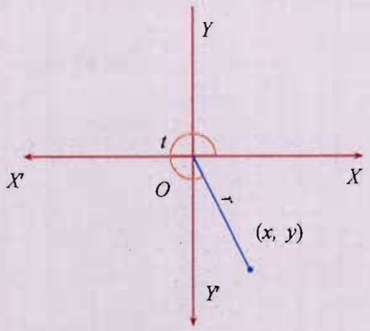
Coordinates and angles

We saw how the trigonometric measures of angles of any size are defined. How do we find them for angles larger than 180°?

First we draw a line through the origin, making this angle with the x-axis:



We then choose a point on this line and find its coordinates and its distance from the origin:



Then by definition,

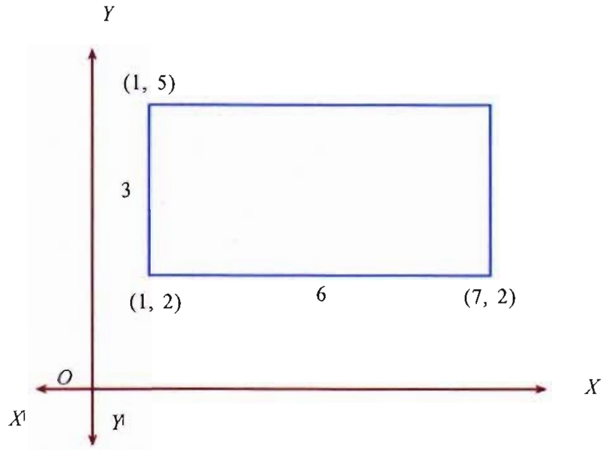
$$\cos t = \frac{x}{r} \quad \sin t = \frac{y}{r}$$

We can also see that

$$\tan t = \frac{y}{x}$$

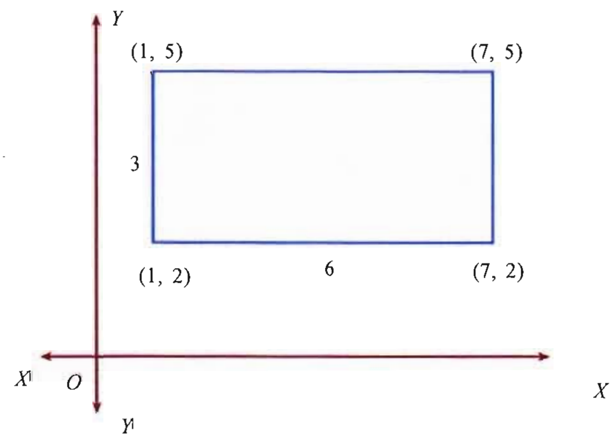
Thus to find the tan measure, we need only the coordinates.

Likewise, what can we say about the general form of the points on the left side of the rectangle? Among these, what are the coordinates of the top-left corner of the rectangle?



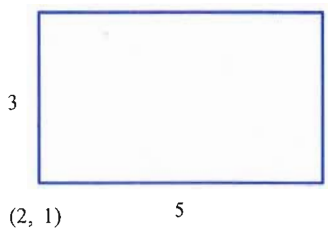
Finally, what about the fourth vertex?

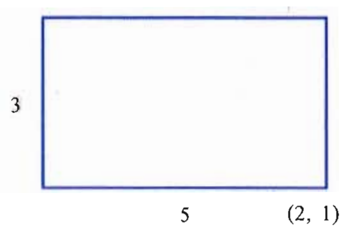
We can think along different lines. Since this point is on the right side of the rectangle, its x-coordinate is 7 (why?) and since it is on the top side of the rectangle also, its y-coordinate is 5.

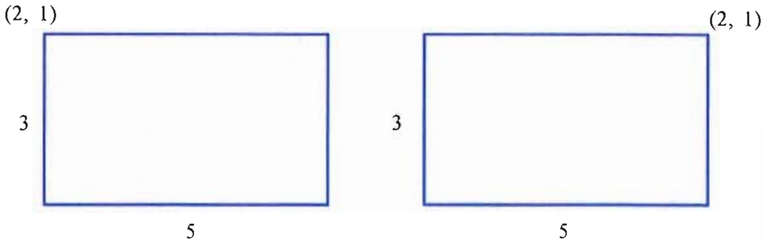


(What are the other ways of getting this?)

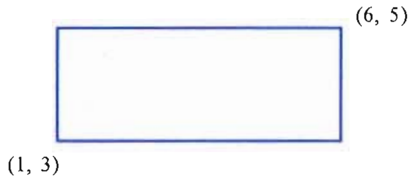
Some rectangles are shown below. The width and height of each is shown; also the coordinate of one vertex with respect to a pair of axes (not shown) parallel to the sides. Find the coordinates of the other vertices.







Now look at this rectangle;



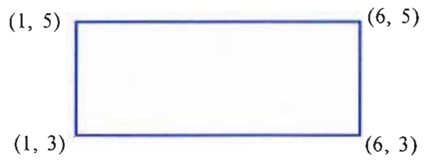
It also has its axes parallel to the axes of coordinates. Can you find out the coordinates of the other vertices?

First let's consider the bottom-right corner. This vertex is on the right side of the rectangle; this side is parallel to the y-axis; one point on it is (6, 5). So, the x-coordinate of the point we seek is also 6.

What about the y-coordinate?

The bottom side is parallel to the x-axis and a point on it is (1, 3). So, the y-coordinate of the bottom-right vertex is also 3.

Similarly, can't we find the top-left vertex also?



Now what are the lengths of the sides of this rectangle?

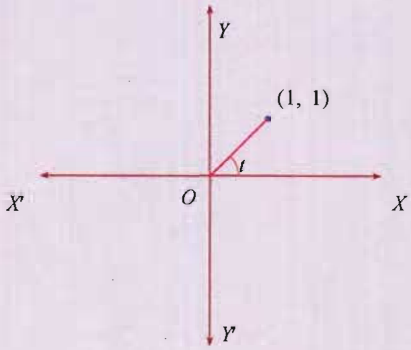
Here's another rectangle with sides parallel to the axes:

Can you find the other vertices? And the lengths of sides?

Can we have a rectangle with any two given points as opposite vertices and sides parallel to the axes?

Lines and points

Can you say how much angle t is in the figure below?



By what we have seen earlier,

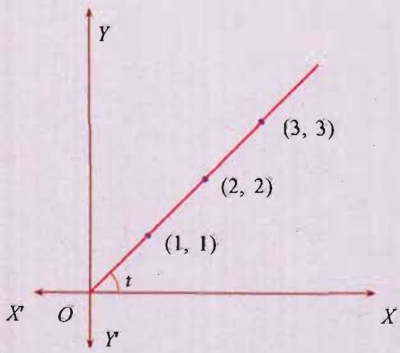
$$\tan t = \frac{1}{1} = 1$$

So,

$$t = 45°$$

We would get the same angle, if we take (2, 2) instead of (1, 1), isn't it so? How about (3, 3)?

So what do we see here? The points (1, 1), (2, 2), (3, 3), . . . are all on the same line.



Have we seen this before? Take another look at the section, **Geometry of arithmetic sequence** in the lesson, **Arithmetic Sequences**.

The line joining these points should not be parallel to either axis, anyhow; that is, their x-coordinates should be different and so must be their y-coordinates.

For example, let's take (–2, 3) and (6, 5). What are the positions of these points?

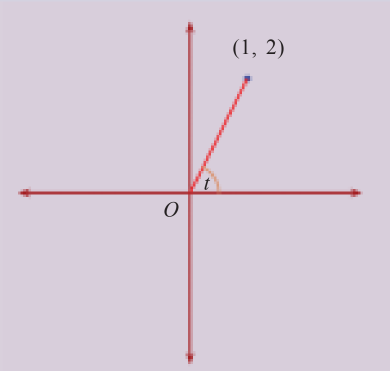
What are the other vertices of the rectangle?

How about (–2, 3) and (6, 1)?

The coordinates of some pairs of points are given below. Without drawing the axes of coordinates, mark these points with the left-right, up-down positions correct. Draw rectangles with these as opposite vertices. Find the coordinates of the other two vertices and the lengths of the sides of these rectangles:

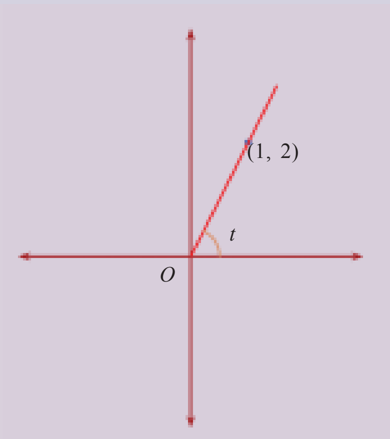
- (3, 5), (7, 8)
- (–3, 5), (–7, 1)
- (6, 2), (5, 4)
- (–1, –2), (–5, -4)

Another line



Can you compute tan t in the figure above?

Suppose we extend this line.



Can you say the coordinates of a few more points on this line?