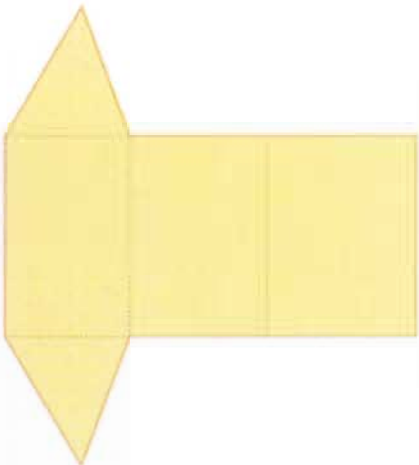
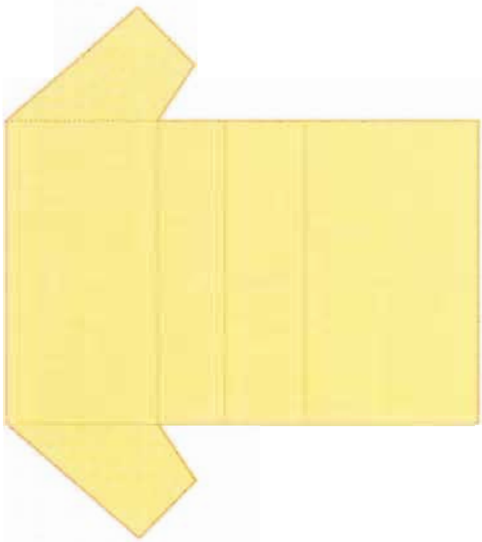
5 Solids

Pyramids

We have seen how we can make various kinds of prisms from paper sheets, by cutting, folding and pasting.





We have also learnt much about prisms.

Now let's make something different. First cut out a figure like the one given below from a thick sheet of paper.

Shapes

Some geometrical shapes such as triangles, rectangles and circles lie flat in a plane; and some such as prisms and cylinders rise up and cannot be confined to a plane.

Prisms and cylinders are seen around us as boxes and blocks and pillars:

We also see solids, which are not prisms.

Pyramids of Egypt

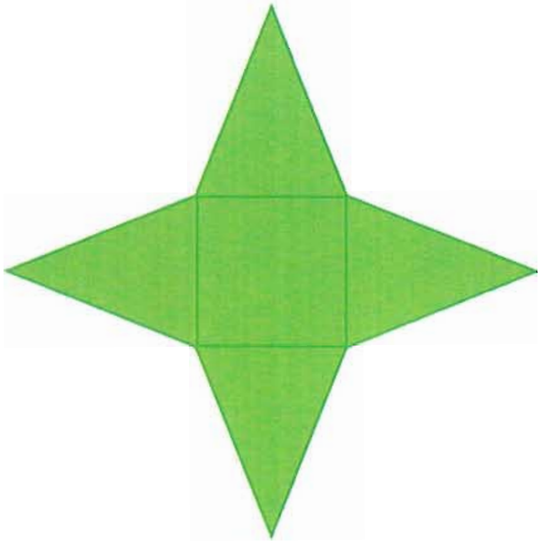
The very name pyramid brings to mind pictures of the great pyramids of Egypt.

About 138 pyramids have been found from various parts of Egypt. Most of them have been built around 2000 BC.

The largest among them is the Great Pyramid of Giza.

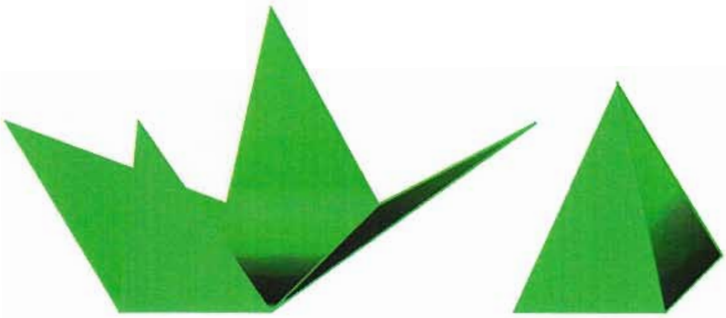
Its base is a square of almost half a lakh square metres; and the height is about 140 metres. It is reckoned that it took almost twenty years to build this.

These royal tombs, starting from a perfect square and rising to a point, erected with gigantic stone blocks, stand as living symbols of human endeavor, engineering skill and mathematical expertise.



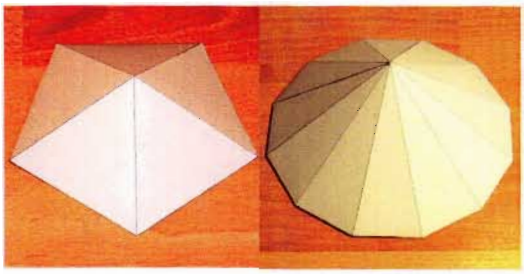
A square in the middle and four triangles around it; and they are (congruent) isosceles triangles.

Fold and paste its edges as shown below:



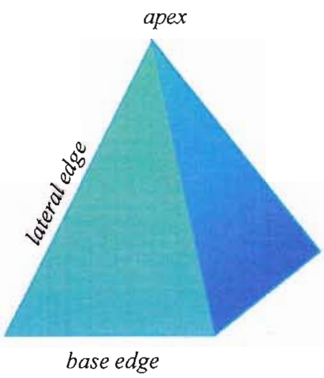
What kind of a solid is this? We cannot call it a prism; a prism has two identical bases and rectangles on its sides. In what we have made, there is a square at the bottom, a point at the top and triangles on the sides.

Instead of a square at the bottom, we can have a rectangle or a triangle or some other polygon. Try! (It looks nicer if the base is a regular polygon.)

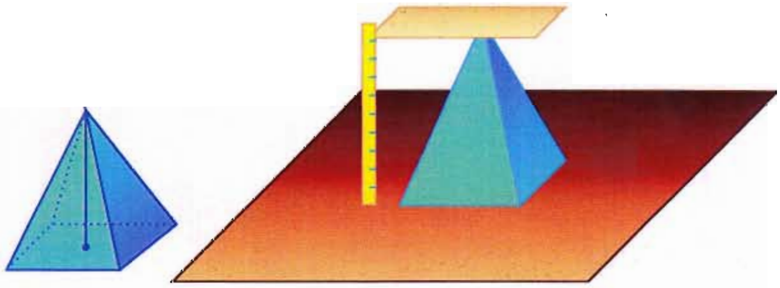


The general name for such a solid is *pyramid*.

The sides of the polygon forming the base of a pyramid are called its *base edges* and the other sides of the triangles are called *lateral edges*. The top end of a pyramid is called its *apex*.



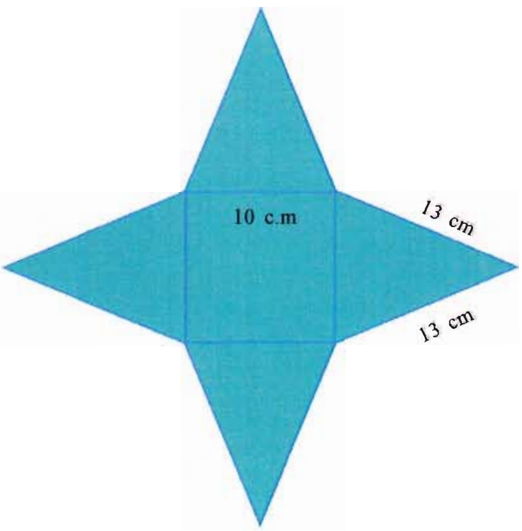
By the height of a prism, we mean the distance between its bases; by the *height* of a pyramid, we mean the perpendicular distance from its apex to its base.



Areas

What is the surface area of a square pyramid with base edges 10 centimetres and lateral edges 13 centimetres?

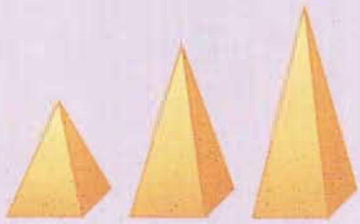
Surface area means the area of the paper needed to make this. How would it look if we cut the pyramid and spread it flat?



Angle and height

To make a square pyramid, we must first fix the base. This also fixes the base of the isosceles triangles on the sides. If we decide on the angle at the top vertex of these triangles also, then the pyramid is completely determined.

As we make this angle smaller, the pyramids become thinner; and we get tall, slender pyramids.



What if we make the angles larger? We get pyramids that are squat and fat:

How large can we make this angle? Can it be 90°?

How large can this angle be for a hexagonal pyramid? And for other pyramids?

We can quickly say that the area of the square in it is 100 square centimetres. What about the triangles?

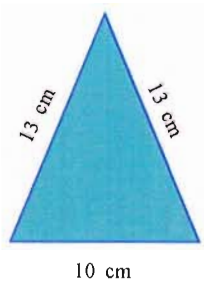
The sides of this triangle are 10, 13 and 13 centimetres. And to find the area using this, Heron comes to our aid: subtract the length of each side from half the perimeter and ...

$$\sqrt{18 \times 8 \times 5 \times 5} = \sqrt{9 \times 16 \times 5 \times 5} = 60$$

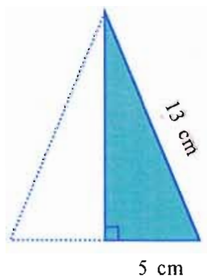
Thus the area of each triangle on the side is 60 square centimetres. So the surface area of the pyramid is

$$100 + (4 \times 60) = 340 \text{ sq.cm}$$

There is another way to compute the area of the triangle.



We need only the height of this triangle, right? Since it is isosceles, the perpendicular from the vertex bisects the base.



Now using Pythagoras Theorem, the length of this perpendicular can be found as

$$\sqrt{13^2 - 5^2} = 12 \text{ cm}$$

And using this, we can find the area of the triangle as $5 \times 12 = 60$ square centimetres.

When the paper is made into a pyramid, what becomes of this height?

Pyramidal numbers

In the section **Triangular numbers,** of the lesson **Square Numbers** in the Class 7 textbook, we saw how dots can be arranged as triangles to get what are called triangular numbers.

In the sequence of numbers we get from this, the n^{th} term is

$$1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1)$$

as seen in the lesson **Arithmetic Sequences.**

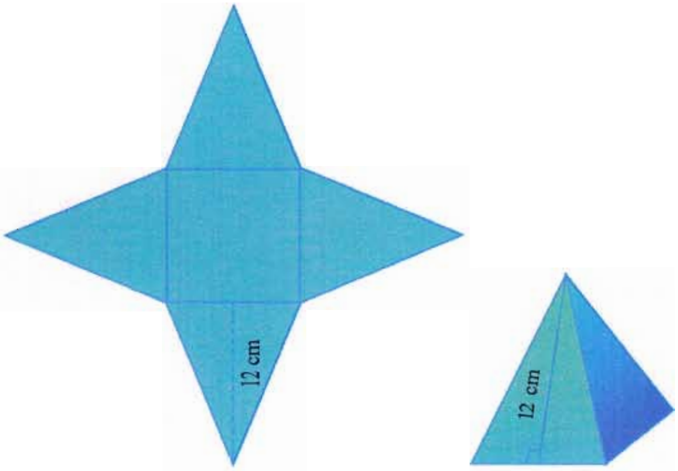
Like this, we can stack small balls in the shape of square pyramids to get a sequence of numbers:

The numbers 1, 5, 14, . . . in this sequence are called *pyramidal numbers*. The n^{th} term of this sequence is

$$1^2 + 2^2 + 3^2 + \ldots + n^2$$

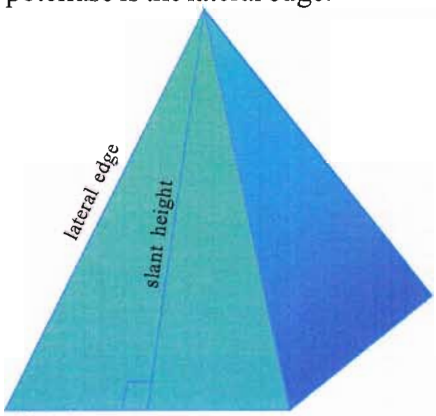
$$= \frac{1}{6}n(n+1)(2n+1)$$

as seen in the section **Sum of squares** of the lesson **Arithmetic Sequences.**



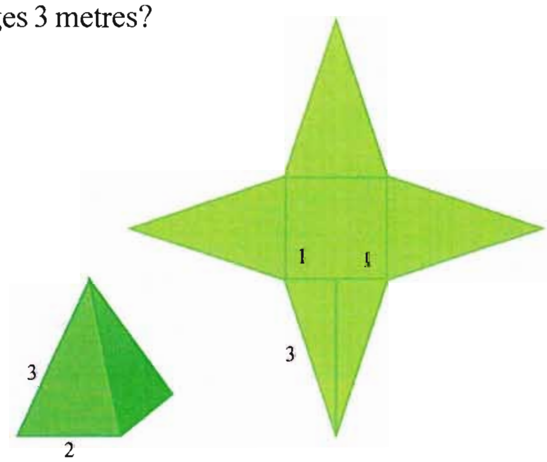
This length is called the *slant height* of the pyramid.

In the problem just done, we saw a relation connecting the base edge, lateral edge and the slant height of a square pyramid. There is a right angled triangle like the one shown below in each face of a square pyramid–its perpendicular sides are half the base edge and the slant height; hypotenuse is the lateral edge.



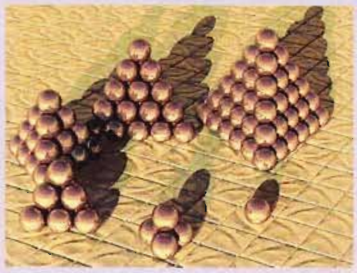
Now can't you do this problem?

What is the surface area of a square pyramid of base edge 2 metres and lateral edges 3 metres?



Tetrahedral numbers

We can also stack small balls as pyramids with equilateral triangle bases.



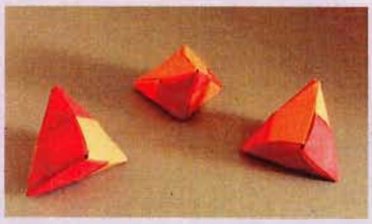
This gives the number sequences 1, 4, 10, ..., that is, each term is a sum of consecutive triangular numbers. It can be proved that the n^{th} term is

$$1 + 3 + 6 + ... + \frac{1}{2}n(n+1)$$

$$= \frac{1}{6}n(n+1)(n+2)$$

Numbers got thus are called *tetrahedral* numbers.

Tetrahedron is the general name for a solid with four triangular faces:



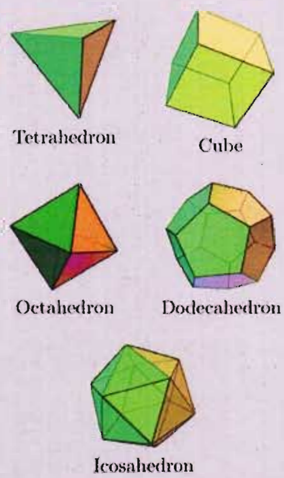
A pyramid based on an equilateral triangle is just a special case of a tetrahedron.

Polyhedrons

We noted that a solid with four triangular faces is called a tetrahedron. The general name for a solid with all its faces polygons is *polyhedron.*

Prisms and pyramids are special cases of polyhedrons. Cylinders and cones are not polyhedrons.

A polyhedron in which the faces are all regular polygons and in which the number of faces coming together at each vertex is the same, is called a *regular polyhedron*. Euclid has proved that there are only five such:



The base area is 4 square metres. To compute the area of the lateral faces, we need the slant height. In the right angled triangle we just referred to, one short side is half the base edge, which is 1 metre and the hypotenuse is the lateral edge, which is 3 metres; so the slant height, which is the third side of this right angled triangle is

$$\sqrt{3^2 - 1^2} = 2\sqrt{2} \text{ m}$$

Using this, the area of each triangular face can be found as

$$\frac{1}{2} \times 2 \times 2\sqrt{2} = 2\sqrt{2} \text{ sq.m}$$

Thus the surface area of the pyramid is

$$4 + (4 \times 2\sqrt{2}) = 4 + 8\sqrt{2} \text{ sq.m}$$

If you are not satisfied with this, then you can use a calculator (or recall an approximate value of $\sqrt{2}$) to compute this as about 15.31 square metres.

Now do these problems on your own:

- A square of side 5 centimetres and four isosceles triangles, each of one side 5 centimetres and the height to the opposite vertex 8 centimetres; these are to be joined to make a square pyramid. How much paper is needed for the job?
- A toy in the shape of a square pyramid has base edge 16 centimetres and slant height 10 centimetres. 500 of these are to be painted and the cost is 80 rupees per square metre. What would be the total cost?
- The lateral faces of a square pyramid are equilateral triangles of side 30 centimetres. What is its surface area?

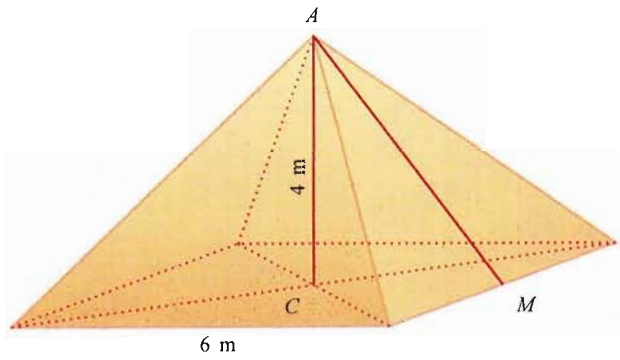
Height and slant height

The height of a pyramid is often an essential measure. See this problem:

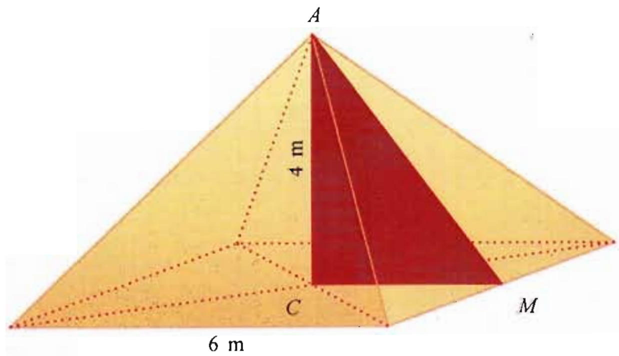
A tent is to be made in the shape of a square pyramid. The sides of its base should be 6 metres and its height should be 4 metres. How much canvas would be needed for this?

To compute the areas of the triangles forming the sides of the tent, we need the slant height. How do we find it from the given facts?

See this picture:



The slant height we need is the length AM. Joining CM, we get a right angled triangle. What is the length of CM?



We can see from the picture that

$$AM = \sqrt{3^2 + 4^2} = 5 \text{ m}$$

So, to make the tent, we need four triangles each of base 6 metres and the height from it 5 metres. Their total area is

$$4 \times \frac{1}{2} \times 6 \times 5 = 60 \text{ sq.m}$$

And this is the area of the canvas needed to make the tent.

What we saw in this problem is true for all square pyramids. Inside any square pyramid, we can visualize a right angled triangle with the slant height as hypotenuse; its perpendicular sides are the height of the pyramid and half the base edge.

Lateral surface area

As in the case of prisms, the sum of the areas of the faces on the sides of a pyramid is called its *lateral surface area.*

If the base of a pyramid is a regular polygon, then the triangles on the sides are all congruent. So, in this case, to find the lateral surface area, we need only multiply the area of one triangle by the number of sides of the base.

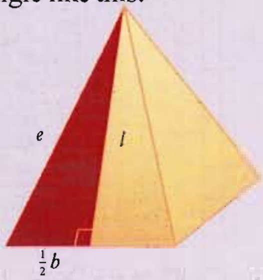
Let's put this in algebra. Suppose the base is a regular polygon of n sides and that each of its sides is of length a. If the slant height of the pyramid is taken as l, then the lateral surface area is

$$n \times \frac{1}{2} \times a \times l$$

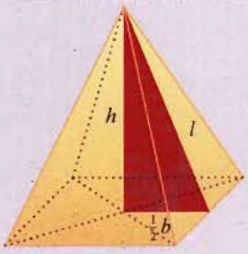
In this, $n \times a$ is the perimeter of the base. Thus the lateral surface area is half the product of the base perimeter and the slant height.

Triangular relations

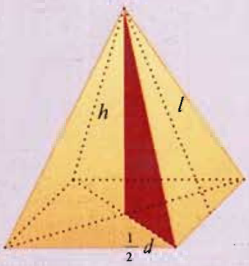
We noted that on each triangular face of a square pyramid, we have a right angled triangle like this:



Also, within the pyramid, another right angled triangle like this:



There's a third right angled triangle within the pyramid, as shown in this picture:



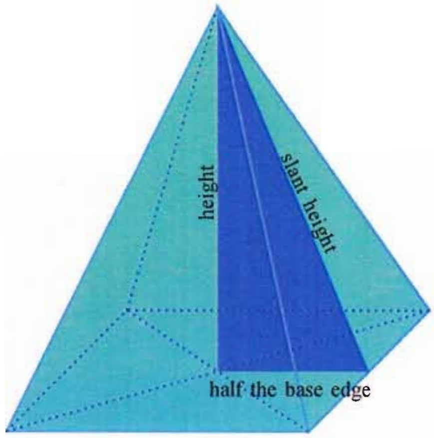
Using these, we get certain relations connecting the length b of the base edge, the length e of the lateral edge, the slant height l, the height h and the length d of the base diagonal:

$$e^2 = l^2 + \frac{1}{4}b^2$$

$$l^2 = h^2 + \frac{1}{4}b^2$$

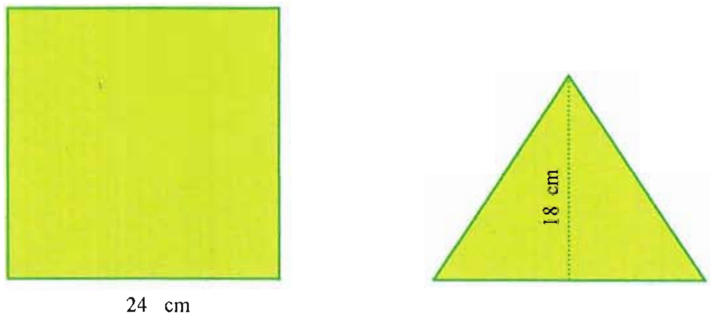
$$e^2 = h^2 + \frac{1}{4}d^2$$

Note that from any two of these equations, we can algebraically derive the third.



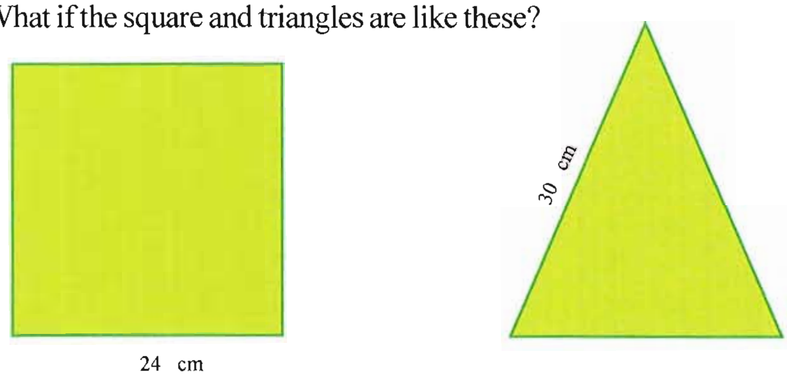
Now can't you do these problems?

- A square pyramid is made using a square and four triangles with dimensions as shown below.



What would be the height of the pyramid?

What if the square and triangles are like these?



- We want to make a paper pyramid with base a square of side 10 centimetres and height 12 centimetres. What should be the lengths of the sides of the triangles?
- Prove that in any square pyramid, the squares of the height, slant height and lateral edge are in arithmetic sequence.

Volume of a pyramid

We have seen that the volume of any prism is the product of its base area and height. What about the volume of a pyramid?

Let's take a square pyramid. First an experiment. Make a square pyramid using stiff paper. Also, a square prism with the same base and height.

Fill the pyramid with sand and pour it into the prism. We would have to do this thrice to fill the prism. Thus we can see that the volume of the pyramid is a third of the volume of the prism. (A mathematical proof of this is given in the **Appendix** at the end of this lesson.)

The volume of the prism is the product of its base area and height. So, what can we say about the volume of the pyramid?

The volume of a square pyramid is a third of the product of its base area and height

For example, the volume of a square pyramid of base edge 10 centimetres and height 8 centimetres is

$$\frac{1}{3} \times 10^2 \times 8 = 266\frac{2}{3} \text{ cu.cm.}$$

Look at this problem:

Each edge of a metal cube is 15 centimetres. It is melted and recast into a square pyramid of base edge 25 centimetres. What would be its height?

The volume of the cube is 15^3 cubic centimetres.

Truncated pyramid

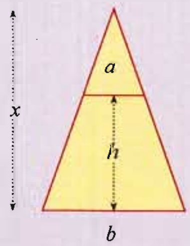
If we cut a square pyramid parallel to its base, we get a smaller square pyramid from the top:

What remains at the bottom?

Such a shape is called a *frustum*; more precisely, here we get a frustum of a square pyramid.

If in such a frustum, we know the lengths of the sides of the squares at the top and bottom and also the height of the frustum, can we compute the height of the original pyramid from which it was cut out?

Look at the triangle got by slicing the whole pyramid vertically through the apex:



From the two similar triangles in this figure, we find

$$\frac{a}{b} = \frac{x-h}{x}$$

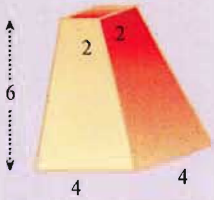
from which we get

$$x = \frac{bh}{b-a}$$

Can you fill in the details?

Volume of a frustum

There is an ancient Egyptian papyrus, dated around BC 1850, in the Pushkin Museum at Moscow. The fourteenth problem in this is to find the volume of a frustum of a square pyramid. Its two square faces are said to have lengths 2 and 4 and the height 6.



The method of computing the volume, given in the papyrus is like this:

Adding the squares of 4 and 2 and twice 4 gives 28; multiplying this by a third of 6, we get 56. This is the volume of the frustum

Let's look at this using algebra, taking a and b for the sides of the squares at the top and bottom, and h for the height. Taking the height of the original pyramid, from which the frustum was cut out as x, its volume would be $\frac{1}{3}b^2x$ and the volume of the smaller pyramid cut out would be $\frac{1}{3}a^2(x - h)$. So that the volume of the frustum is $\frac{1}{3}(b^2 - a^2(x - h))$

In this, we have already seen that

$$x = \frac{bh}{b - a}$$

Using this in the expression above and simplifying, we get the volume as

$$\frac{1}{3}h(b^2 + ab + b^2)$$

But this is precisely what the papyrus says.

This is also the volume of the pyramid made. We know that the volume of the pyramid is a third of the product of its height and base area. Since the base area of the pyramid is given to be 25^2 square centimetres, we can find a third of the height as $\frac{15^3}{25^2}$ and then the height as

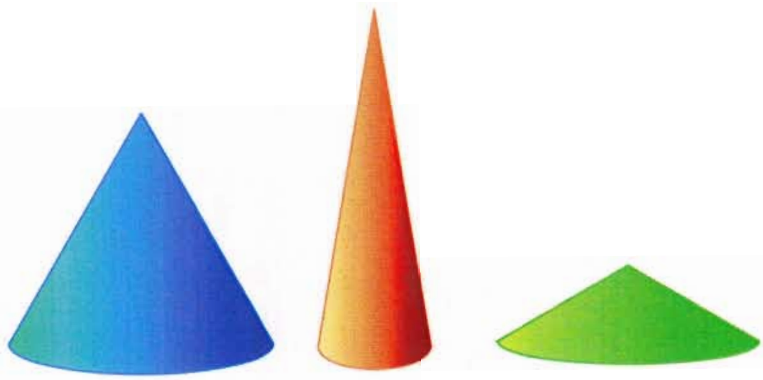
$$3 \times \frac{15^3}{25^2} = 16.2 \text{ cm}$$

Now you can try some problems:

- What is the volume of a square pyramid of base edge 10 centimetres and slant height 15 centimetres?
- In two square pyramids of the same volume, the base edge of one is half the base edge of the other. How many times the height of the pyramid with larger base is the height of the other?
- The base edges of two square pyramids are in the ratio 1 : 2 and their heights are in the ratio 1 : 3. The volume of the first pyramid is 180 cubic centimetres. What is the volume of the second?

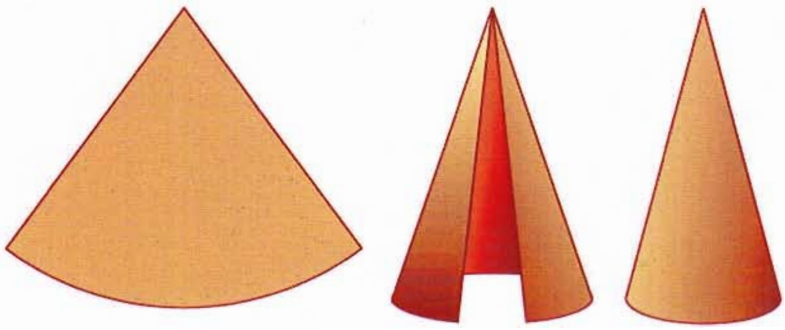
Cones

Like cylinders, which can be thought of as prisms with circular base, we have solids like these, which can be thought of as pyramids with circular base.

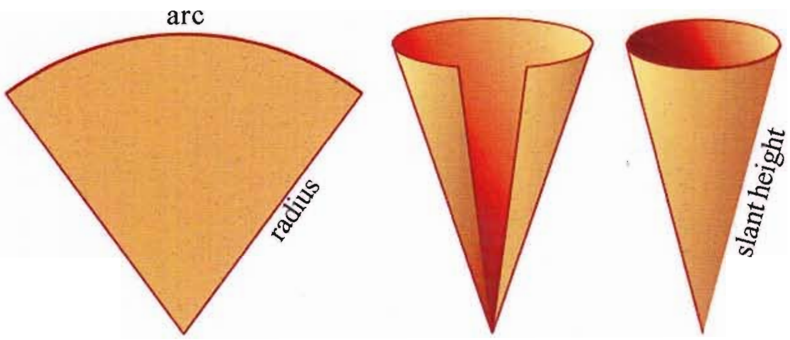


Such a solid is called a *cone*.

Just as we can make a cylinder by rolling up a rectangle, we can make a cone by rolling up a sector of a circle. (If we want a closed cone, then we need another small circle also.)



In this, what's the relation between the dimensions of the sector rolled up and those of the cone made?



The radius of the sector becomes the slant height of the cone; the arc length of the sector becomes the base-circumference of the cone.

We often specify the size of a sector in terms of its central angle. See this problem:

A sector of central angle 45° is cut out from a circle of radius 12 centimetres, and it is rolled into a cone. What is the radius of the base of this cone and what is its slant height?

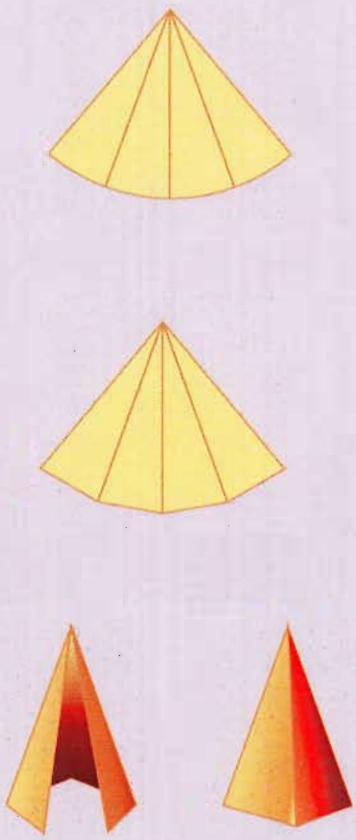
The slant height of the cone is just the radius of the circle, which is given to be 12 centimetres. What about the radius of the base?

The central angle 45° is $\frac{1}{8}$ of 360°; and the length of the arc of a sector is proportional to the central angle. So, the arc length of this sector is $\frac{1}{8}$ of the circumference of the full circle.

The arc of the circle becomes the circumference of the base circle of the cone. Thus the circumference of the base circle of the cone is $\frac{1}{8}$ the circumference of the large circle from which the sector is cut

Sectors and pyramids

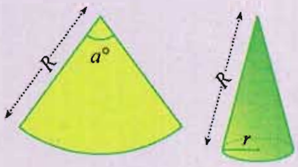
We cannot make a cone by pasting triangles around a circle, as we do for pyramids; but we can make pyramids by bending a sector as we do for cones. See how we make a square pyramid.



By dividing the sector into more equal parts, can't we make other pyramids also?

Radius and slant height

Suppose a sector of radius R and central angle $a°$ is rolled into a cone.



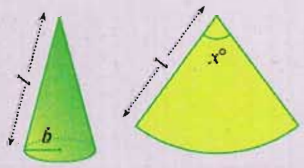
Then the slant height of the cone is R. To find the base-radius, note that the arc-length of the sector is $\frac{a}{360} \times 2\pi R$; and this is the circumference of the base of the cone. So, if we take the base-radius of the cone as r, then

$$2\pi r = \frac{a}{360} \times 2\pi R$$

and this gives

$$r = \frac{a}{360} \times R$$

On the other hand, suppose that a cone of base-radius b and slant height l is cut open and spread flat as a sector.



The radius of the sector is l. And if we take the central angle as $x°$, then we get

$$\frac{x}{360} \times 2\pi l = 2\pi b$$

and from this

$$x = \frac{b}{l} \times 360$$

out. Since the radius of a circle is proportional to the circumference, the radius of the small circle is $\frac{1}{8}$ times the radius of the large circle.

Thus the radius of the base of the cone is $\frac{1}{8} \times 12 = 1.5$ centimetres.

How about a question in reverse?

How do we make a cone of base radius 5 centimetres and slant height 15 centimetres?

To make a cone, we need a sector of a circle. Since the slant height is to be 15 centimetres, the sector must be cut out from a circle of radius 15 centimetres. What about its central angle?

The radius of the small circle forming the base of the cone is to be $\frac{5}{15} = \frac{1}{3}$ of the radius of the large circle from which the sector is to be cut out. So, the circumference of the small circle is also $\frac{1}{3}$ of the circumference of the large circle.

The circumference of the small circle is the arc length of the sector.

Thus the arc of the sector should be $\frac{1}{3}$ of the circle from which it is cut out. So, its central angle must be $360 \times \frac{1}{3} = 120°$.

Now try these problems:

- What is the base-radius and slant height of the cone made by rolling up a sector of radius 10 centimetres and central angle 60°?
- What is the central angle of the sector needed to make a cone of base-radius 10 centimetres and slant height 25 centimetres?
- What is the ratio of the base-radius and slant height of a cone made by rolling up a semicircle?

Curved surface area

Like a cylinder, a cone also has a *curved surface;* the slanted surface rising up.

The area of this curved surface is the area of the sector used to make the cone. (In the case of a cylinder also, the curved surface area is the area of rectangle rolled up to make it, isn't it?)

Look at this problem:

What is the area of the paper needed to make a conical hat of base-radius 8 centimetres and slant height 30 centimetres?

What we need is the area of the circular sector needed to make such a hat. Since the slant height is to be 30 centimetres, the sector must be cut out from a circle of this radius.

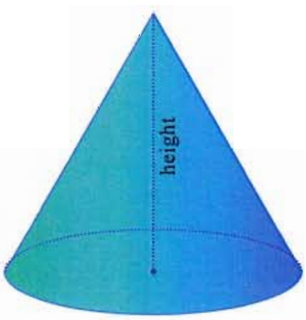
Also, the radius of the small circle forming the base of the cone is to be 8 centimetres; that is, $\frac{8}{30} = \frac{4}{15}$ of the radius of the large circle from which the sector is to be cut out. So, the circumference of the small circle is the same fraction of the circumference of the large circle. But the circumference of the small circle is the arc-length of the sector.

Thus the sector must be $\frac{4}{15}$ of the full circle. So, its area is also this fraction of the area of the circle; that is

$$\pi \times 30^2 \times \frac{4}{15} = \pi \times 2 \times 30 \times 4 = 240\pi$$

So, to make the hat, we need 240π square centimetres of paper. (Doing further computations, we can get this as about 754 square centimetres.)

As in a square pyramid, the *height* of a cone is the perpendicular distance from the apex to the base; and it is the distance between the centre of the base-circle and the apex.



Curved surface area

The curved surface area of a cone is the area of the sector used to make it. If we take the base-radius of the cone as r and the slant height as l, then the radius of the sector is l and its central angle is $\frac{r}{l} \times 360$.

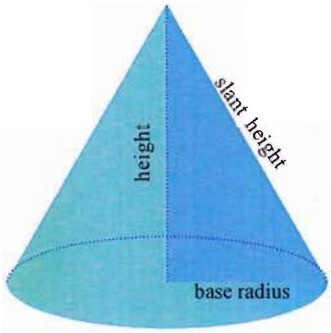
So, its area can be computed as

$$\frac{1}{360} \times \left(\frac{r}{l} \times 360\right) \times \pi l^2 = \pi r l$$

(Recall the computation of the area of a sector, done in Class 9)

Thus the curved surface area of a cone is half the product of the base circumference and the slant height.

Again, as in a square pyramid, there is a relation between base-radius, height and slant height in a cone also, effected by a right angled triangle.



For example, in a cone of base-radius 5 centimetres and height 10 centimetres, the slant height is $\sqrt{5^2+10^2} = 5\sqrt{5}$ centimetres.

Now some problems for you:

- What is the curved surface area of a cone of base radius 12 centimetres and slant height 25 centimetres?
- What is the curved surface area of a cone of base diameter 30 centimetres and height 40 centimetres?
- A cone shaped firework is of base-diameter 10 centimetres and height 12 centimetres. 10000 such fireworks are to be wrapped in colour paper. The price of paper is 2 rupees per square metre. What is the total cost of wrapping?
- Prove that for a cone made by bending a semicircle, the curved surface area is double the base area.

Volume of a cone

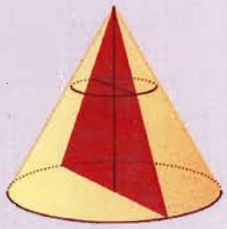
Remember the experiment done to find out the volume of a square pyramid? We can do a similar experiment with a cone and a cylinder of the same base–radius and height. And see that the volume of the cone is a third of the volume of the cylinder. That is

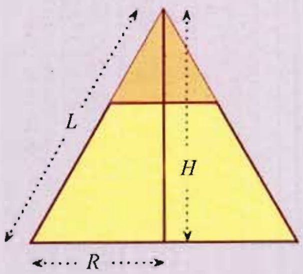
The volume of a cone is a third of the product of its base area and height

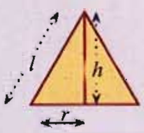
Small and large

If we slice a cone parallel to its base, we get a small cone from the top:

Is there any relation between the dimensions of this small cone and the original large cone?







If we denote the base-radius, height and slant height of the large cone as R, H, L and those of the small cone as r, h, l then from the pictures above, we can see that

$$\frac{r}{R} = \frac{h}{H} = \frac{l}{L}$$

(A mathematical proof of this also is given in the **Appendix** at the end of the lesson.)

For example, the volume of a cone with base-radius 4 centimetres and height 6 centimetres is

$$\frac{1}{3} \times \pi \times 4^2 \times 6 = 32\pi \text{ cu.cm.}$$

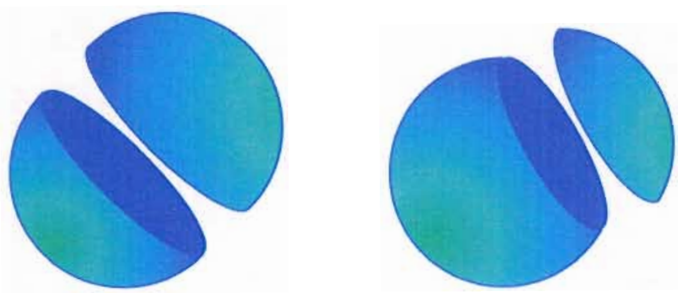
The problems below are for you:

- The base-radius of a cylindrical block of wood is 15 centimetres and its height is 40 centimetres. What is the volume of the largest cone that can be carved out from this?
- A solid metal cylinder is of base-radius 12 centimetres and height 20 centimeters. By melting and recasting, how many cones of base-radius 4 centimetres and height 5 centimetres can be made?
- A sector of central angle 216° is cut out from a circle of radius 25 centimetres and it is rolled up into a cone. What is the base-radius and height of this cone? What is its volume?
- The ratio of the base-radii of two cones is 3 : 5 and their heights are in the ratio 2 : 3. What is the ratio of their volumes?
- Two cones have the same volume and their base - radii are in the ratio 4 : 5. What is the ratio of their heights?

Spheres

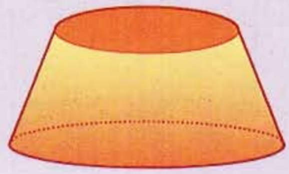
We have watched with thrill the soaring flight of a football or a cricket balls and tasted with relish small round sweets like *laddus*. Now let's look at the mathematics of such round things little or large, called *spheres*.

If we slice a cylinder or cone parallel to its base, we get a circle. In whatever way we slice a sphere, we get a circle.

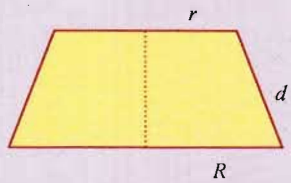


Frustum of a cone

Suppose we slice a cone parallel to its base and cut off a small cone from the top. The part remaining at the bottom is called a *frustum of a cone.*



If we know the radii of the top and bottom circles of such a frustum and also its slant height, can we compute its curved surface area?



If the slant heights of the original large cone and the small cone cut off are denoted L and l, then $d = L - l$ in the figure above. So, the curved surface area of the frustum is

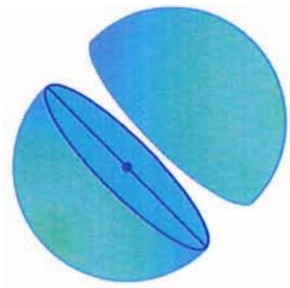
$$\begin{aligned}\pi RL - \pi rl &= \pi(RL - rl)\\ &= \pi\big(R(l+d) - rl\big)\\ &= \pi(Rl + Rd - rl)\end{aligned}$$

Also, we have seen earlier that $\frac{r}{R} = \frac{l}{L}$, which gives $Rl = rL$. Using this, the expression for the curved surface area becomes

$$\begin{aligned}\pi(rL + Rd - rl) &= \pi\big(r(L - l) + Rd\big)\\ &= \pi(rd + Rd)\\ &= \pi(r + R)d\end{aligned}$$

The distance of any point on a circle from the centre of the circle is the same, isn't it? A sphere also has a centre; and the distance of any point on the sphere from this centre is the same. This distance is called the radius of the sphere and double the radius is called its diameter.

If we cut a sphere into two identical halves, we get a circle; and the centre, radius and diameter of this circle are the centre, radius and diameter of the sphere.



We cannot cut open a sphere and spread it flat to compute its area, as with other solids. The fact is that we cannot cut and spread out a sphere without some folds or without some stretching.

But we can prove that the surface area of a sphere of radius r is $4\pi r^2$. (This is given in the **Appendix** at the end of this lesson.)

In other words,

The surface area of a sphere is the product of the square of its radius by 4π.

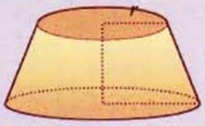
Also, we can prove that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. (This also is given in the **Appendix**.)

Let's look at some examples:

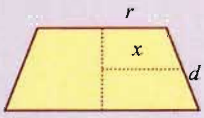
- What is the surface area of the largest sphere that can be carved out from a cube of side 8 centimetres?

Frustum and cylinder

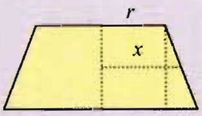
As we have seen, the curved surface area of the frustum shown below is $\pi(r+R)d$



Let's denote by x, the radius of the circle at the middle, as shown in the figure below:



Let's draw one more line as shown below:



From the two similar right angled triangles on the right, we get

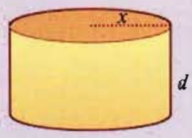
$$\frac{x-r}{R-r}=\frac{1}{2}$$

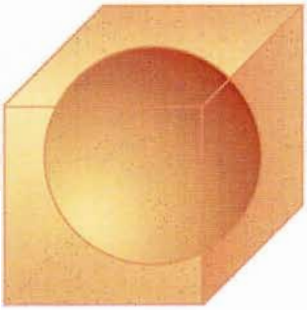
Simplifying this, we get

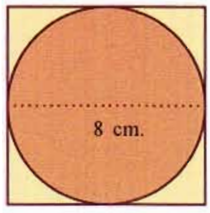
$$x=\frac{1}{2}(R+r)$$

So, the curved surface area of the frustum is $2\pi xd$.

But this is the curved surface area of the cylinder with base-radius x and height d, isn't it?







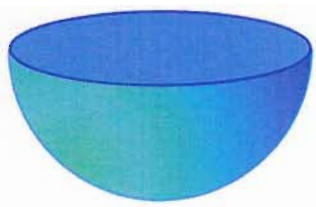
From the pictures above, we see that the diameter of the sphere is the length of an edge of the cube. So, the surface area of the sphere is

$$4\pi \times 4^2 = 64\pi \text{ sq.cm.}$$

Let's look at another problem

- A solid sphere of radius 12 centimetres is cut into two equal halves. What is the surface area of the hemisphere so got?

The surface of a hemisphere consists of half the surface of the sphere and a circle.



Since the radius of the sphere is 12 centimetres, its surface area is

$$4\pi \times 12^2 = 576\pi \text{ sq.cm}$$

The surface area of the hemisphere is half of this added to the area of the circle. Since the radius of the circle is also 12 centimetres, its area is

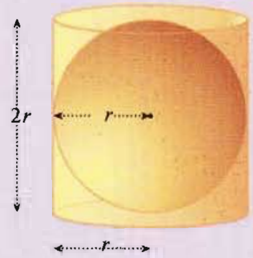
$$\pi \times 12^2 = 144\pi \text{ sq.cm.}$$

Thus, the surface area of the hemisphere is

$$\frac{1}{2} \times 576\pi + 144\pi = 432\pi \text{ sq.cm.}$$

Sphere and cylinder

Consider a cylinder which just covers a sphere:



The base-radius of the cylinder is the radius of the sphere and the height of the cylinder is the diameter of the sphere.

That is, if we denote the radius of the sphere by r, then the base - radius of the cylinder is also r and the height of the cylinder is $2r$. So, the surface area of the cylinder is

$$(2\pi r \times 2r) + (2 \times \pi r^2) = 6\pi r^2$$

The surface area of the sphere is $4\pi r^2$. Thus the ratio of these surface areas is 3 : 2

Again, the volume of the cylinder is

$$\pi r^2 \times 2r = 2\pi r^3$$

and the volume of the sphere is $\frac{4}{3}\pi r^3$. So, the ratio of the volumes is also 3 : 2.

Thus we see that ratio of the surface areas and the ratio of the volumes of a sphere and the cylinder which just covers it, are both equal to 3 : 2.

Archimedes

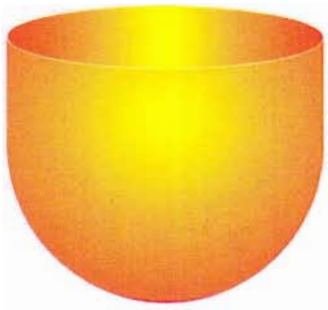
It was the great Archimedes who discovered the fact that a sphere and the cylinder just covering it has the ratio 3 : 2 for surface areas and volumes. It is said that this idea was so dear to him that he asked for this to be engraved on his tomb.

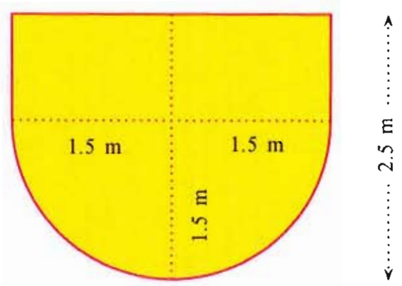
We have seen in Class 8, the story of Archimedes's defence of his home country Syracuse against the invading Roman army. The Romans did conquer Syracuse in BC 212. A soldier, whose name no one remembers, killed Archimedes.

About a hundred and fifty years later, the Roman scholar Cicero, discovered Archimedes's tomb. What helped him was the sight of a stone cylinder and sphere rising above thorns and bush. As an act of penitence, Cicero had the area cleared and paid his respects to one of the greatest scientists ever.

One more example:

A water tank is in the shape of a hemisphere attached to the bottom of a cylinder. Its radius is 1.5 metres and total height is 2.5 metres. How many litres of water can it contain?





The volume of the hemispherical part of the tank is

$$\frac{2}{3}\pi \times 1.5^3 = 2.25\pi \text{ cu.m}$$

and the volume of the cylindrical part is

$$\pi \times 1.5^2(2.5 - 1.5) = 2.25\pi \text{ cu.m}$$

So, the total volume of the tank is

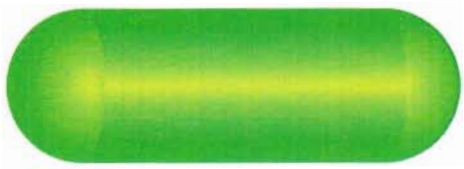
$$2.25\pi + 2.25\pi = 4.5\pi \approx 14.13 \text{ cu.m}$$

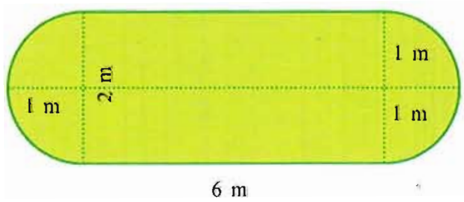
Since a cubic metre is 1000 litres, the tank can contain about 14130 litres.

Now some problems for you:

- The volumes of two spheres are in the ratio 27 : 64. What is the ratio of their radii?

- A metal cylinder of base-radius 4 centimetres and height 10 centimetres is melted and recast into spheres of radius 2 centimetres. How many such spheres are got?
- The picture below shows a petrol tank.





How many litres does it hold?

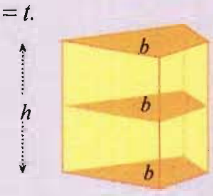
Catch-all

The formulas to compute the volumes of prisms, pyramids and sphere are all different, aren't they? But there is a single formula which works for all these.

Let b be the area of the bottom, m be the area at the middle, t be the area at the top and h be the height of any of these kinds of solids. Then the volume is

$$\frac{1}{6}h(b+4m+t)$$

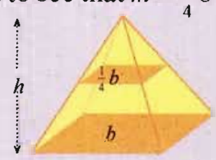
For prisms, the area at the bottom, middle and top are all the same; that is $b = m = t$.



So, by this formula, the volume of a prism is

$$\frac{1}{6}h(b+4b+b) = \frac{1}{6}h \times 6b = bh$$

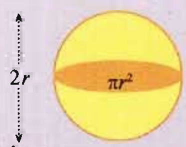
What about pyramids? It is not difficult to see that $m = \frac{1}{4}b$ and $t = 0$



So, the volume of a pyramid is

$$\frac{1}{6}h(b+b+0) = \frac{1}{6}h \times 2b = \frac{1}{3}bh$$

Now what about a sphere? If we take the radius as r, then $m = \pi r^2$, $b = t = 0$, $h = 2r$



So, the volume is

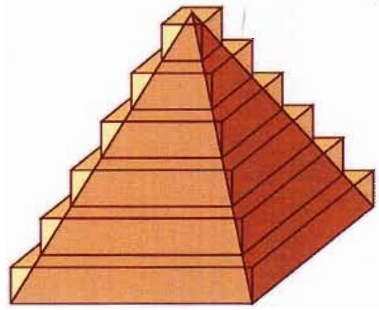
$$\frac{1}{6} \times 2r \times 4\pi r^2 = \frac{4}{3}\pi r^3$$

Appendix

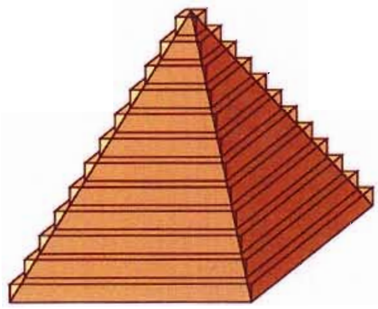
In the lesson, we have mentioned only the techniques of finding the volume of pyramids and cones, as also the surface area and volume of spheres. For those who are interested in knowing how these formulas are actually got, we give below the proofs.

Volume of a square pyramid

We can think of a stack of square plates of decreasing size as an approximation to a square pyramid.



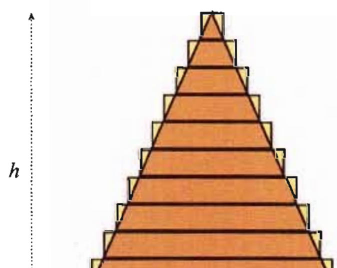
As we decrease the thickness of plates and increase their number, we get better approximations.



So, as we do this, the sum of the volumes of these plates give better and better approximations to the volume of the pyramid.

To start with, suppose we use 10 plates. Each plate is a short square prism. Let's take them to be of equal height. So, if the height of the pyramid we are trying to approximate is h, then the height of each plate is $\frac{1}{10}h$. Now how do we find the base-edge of each plate?

If we slice the pyramid and the enveloping plates vertically through the apex, we get a figure like this:



Starting from the top, we have a sequence of isosceles triangles of increasing size. Their heights increase at the rate of $\frac{1}{10}h$ for each plate. These triangles are all similar (why?) and so their bases also increase at the same rate. Thus if we take the base-edge of the pyramid as b, the bases of the triangles, starting at the top, are $\frac{1}{10}b, \frac{2}{10}b, \dots, b$

So, the volumes of these plates are

$$\left(\frac{1}{10}b\right)^2 \times \frac{1}{10}h, \left(\frac{2}{10}b\right)^2 \times \frac{1}{10}h, \dots b^2 \times \frac{1}{10}h$$

What about their sum?

$$\frac{1}{10}b^2h\left(\frac{1}{10^2}+\frac{2^2}{10^2}+\dots+\frac{9^2}{10^2}+\frac{10^2}{10^2}\right) = \frac{1}{1000}b^2h\left(1^2+2^2+3^2+\dots+10^2\right)$$

We have seen how such sums can be computed in the section **Sum of squares** of the lesson **Arithmetic Sequence**.

$$1^2 + 2^2 + 3^2 + \dots + 10^2 = \frac{1}{6} \times 10 \times (10+1) \times (2 \times 10 + 1)$$

So, the sum of the volumes is

$$\frac{1}{1000}b^2h \times \frac{1}{6} \times 10 \times 11 \times 21 = \frac{1}{6}b^2h \times \frac{10}{10} \times \frac{11}{10} \times \frac{21}{10} = \frac{1}{6}b^2h \times 1.1 \times 2.1$$

Now imagine of 100 such plates. (We cannot draw it anyway.)

The thickness of the plates become $\frac{1}{100}h$; the base-edges become $\frac{1}{100}b, \frac{2}{100}b, \frac{3}{100}b, \dots$, and the sum of the volumes would be

$$\frac{1}{100^3}b^2h\left(1^2+2^2+3^2+\dots+100^2\right) = \frac{1}{100^3}b^2h \times \frac{1}{6} \times 100 \times 101 \times 201$$

$$= \frac{1}{6}b^2h \times \frac{100}{100} \times \frac{101}{100} \times \frac{201}{100}$$

$$= \frac{1}{6}b^2h \times 1.01 \times 2.01$$

What if the number of plates is made 1000? Without actually computing, we can see that it is

$$\frac{1}{6}b^2h \times 1.001 \times 2.001$$

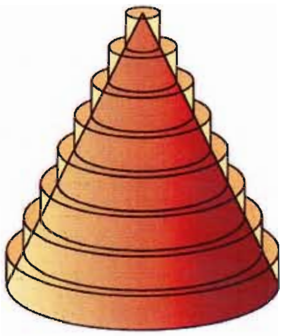
What is the number to which these numbers get closer and closer?

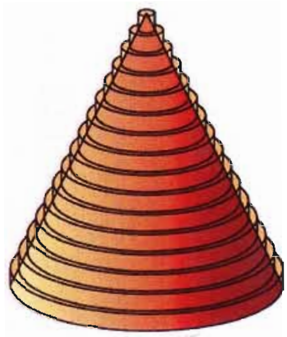
It is the volume of the pyramid. That is,

$$\frac{1}{6}b^2h \times 1 \times 2 = \frac{1}{3}b^2h$$

Volume of a cone

Just as we stacked square plates to approximate a pyramid, we can stack circular plates to approximate a cone.

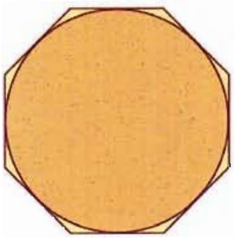




And in much the same way, we can compute the volume of the cone also. (Try!)

Surface area of a sphere

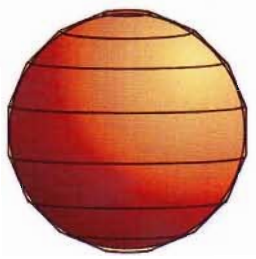
First consider a circle through the middle of the sphere and a polygon which just contains it. (In more mathematical language, our circle is the incircle of the polygon.)



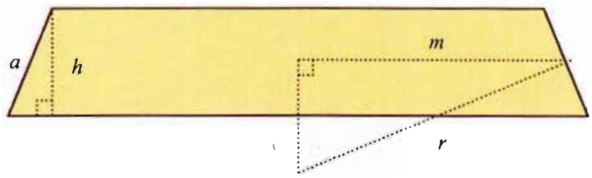
Now if we think of a rotation of these, then we get a sphere and a solid which just covers it.

In the figure above, the solid covering the sphere can be divided into two frustums of cones and a cylinder in the middle.

As the number of sides of the polygon covering the circle increases, the solid generated by rotation approximates the sphere better.



To find the surface area of these frustums, let's consider one of them separately. Let's take the radius of its middle circle as m and its height as h. If we also take the radius of the sphere as r and the length of a side of the covering polygon as a, we get a figure like this, showing all these:



Since the two right angled triangles in it are similar, we get

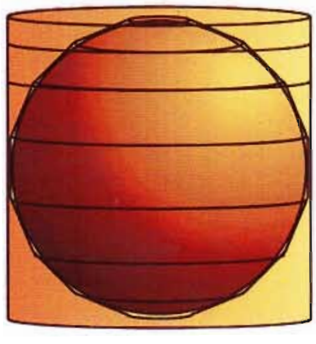
$$\frac{m}{r} = \frac{h}{a}$$

from which we get

$$am = rh$$

The surface area of the frustum got by rotating this is $2\pi ma$, as seen in the section, **Frustum and cylinder**. Using the equation above, we see that this also equal to $2\pi rh$; that is the curved surface area of a cylinder with base - radius r and height h.

So, what do we find? In the solid approximating the sphere, the surface area of each frustum is equal to the surface area of a cylinder with the same height and base-radius equal to that of the sphere. So, the surface area of this solid is the sum of the surface areas of all these cylinders. And what do we get on putting together all these cylinders? A large cylinder that just covers the sphere.



As the number of sides of the covering polygon circle increases, it becomes more and more circle like; and the solid got by rotation becomes more and more sphere like. As seen just now, the surface area of any such shape is equal to the surface area of the cylinder just covering the sphere. So, the surface area of the sphere is also equal to the surface area of the enveloping cylinder. Since the base-radius of the cylinder is r and its height is $2r$, its surface area is

$$2\pi \times r \times 2r = 4\pi r^2$$

and this is the surface area of the sphere also.

Volume of a sphere

See this picture:

The sphere is divided into cells by means of horizontal and vertical circles. If we join the corners of such a cell with the centre of the sphere, we get a pyramid like solid.

The sphere is the combination of these solids and so the volume of the sphere is the sum of the volumes of these solids. Now if we change each of these cells into an actual square which touches the sphere, then we get a solid which just covers the sphere; and this solid is made up of actual square pyramids. All these pyramids have height equal to the radius r of the sphere. If we take the base area of each such pyramid as a, then its volume is $\frac{1}{3}ar$. The volume of the solid covering the sphere is the sum of the volumes of these pyramids. The bases of all the pyramids together make up the surface of this solid. So, if we take the surface area of this solid as s, then its volume is $\frac{1}{3}sr$.

As the size of the cells decreases and their number increases, the solid covering the sphere gets closer to the sphere; and its surface area s to the surface area of the sphere, which is $4\pi r^2$. Thus the volume of the sphere is

$$\frac{1}{3} \times 4\pi r^2 \times r = \frac{4}{3}\pi r^3$$