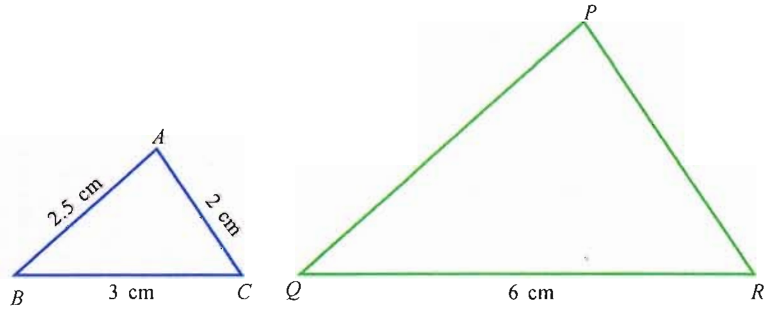
4 Trigonometry

Sides and angles

Look at these two triangles:



ΔPQR has the same angles as ΔABC. More precisely,

$$\angle P = \angle A \qquad \angle Q = \angle B \qquad \angle R = \angle C$$

And QR is twice as long as BC.

What can you say about the lengths of the other sides of ΔPQR?

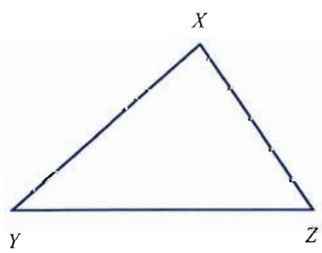
The side opposite $\angle P$ is twice as long as the side opposite $\angle A$; and the sides opposite the other equal pairs of angles must also have the same ratio, right?

In other words,

$$PQ = 2 \times AB = 5 \text{ cm}$$

$$RP = 2 \times CA = 4 \text{ cm}$$

Now look at this triangle:



In this also, we have

$$\angle X = \angle A \qquad \angle Y = \angle B \qquad \angle Z = \angle C$$

On the earth and up the sky

Trigonometry is the study of the relations between the sides and angles of triangles.

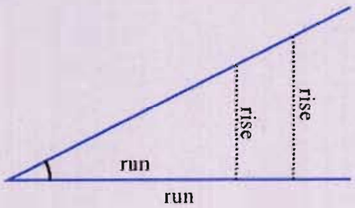
We have seen that angles are used to measure slant or spread or turn. (The section **Slant and spread and turn,** of the lesson **Circle measures** in the Class 9 textbook.) In history, we first come across measures of slant in various constructions on the earth; and measures of turn in the study of planets in the sky.

The first astronomical studies were also for earthly needs. The basic need of man is food; and production of food, that is agriculture, depends on the weather. A factor influencing weather is the revolution of the earth around the sun. To understand this well, we must be able to determine the positions of the other planets and the stars. This is why in all ancient agricultural communities, astronomy was a major topic of study. And mathematics, especially geometry, is very much necessary for this.

Measure of slant

We have already seen the Babylonian technique of measuring angles by dividing a circle into 360 equal parts and its connection with astronomy. (The section **History of angle measurement,** of the lesson **Slant and spread** in the Class 6 textbook.) This method was used in Babylon from the third century BC. And this is the degree measure that we now use.

But in the constructions on the earth, another method was used to measure slants. See this picture:



As shown in the picture, the "run" and "rise" change from one position to another, but if we divide the rise by the run at any position, we get the same number. (Why?) And this number differs from angle to angle, depending on its size. It is this number, which is used as a measure of slant.

In the Ahmose Papyrus from ancient Egypt (see the section **Ancient Math,** of the lesson **Equations** in the Class 8 textbook), we can see such computations. The slant between the triangular faces and the base of square pyramid is computed in this manner.

In a clay tablet from ancient Babylon, we see tables of numbers got by dividing the hypotenuse by another side, for various right angled triangles.

Without knowing the length of at least one side, what can we say?

At least this much:

$$\frac{XY}{2.5} = \frac{YZ}{3} = \frac{ZX}{2}$$

Getting rid of decimals (or using ΔPQR instead of ΔABC), we can write this as

$$\frac{XY}{5} = \frac{YZ}{6} = \frac{ZX}{4}$$

From this, we can write

$$\frac{XY}{YZ} = \frac{5}{6}; \quad \frac{YZ}{ZX} = \frac{6}{4}$$

or combine all these into the single equation

$$XY : YZ : ZX = 5 : 6 : 4$$

The lengths of the sides of the other two triangles also bear the same ratio, don't they?

What general principle do we see here? We can draw several triangles with the same three angles; and the actual lengths of the sides may vary from triangle to triangle. But the ratio of these lengths doesn't change.

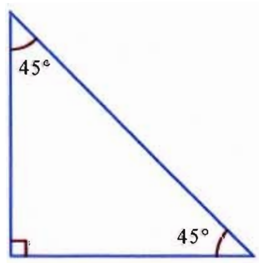
In short,

For triangles with the same set of angles, the ratio of the lengths of the sides is the same.

This leads to another thought: if the angles of a triangle are known, can we find the ratio of their sides?

Let's look at a couple of examples:

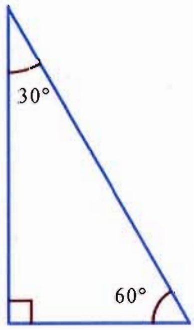
- See this triangle:



The perpendicular sides of this triangle are equal (Reason?) And if this length is taken as x, then the hypotenuse should be $\sqrt{2}\,x$. (Why?)

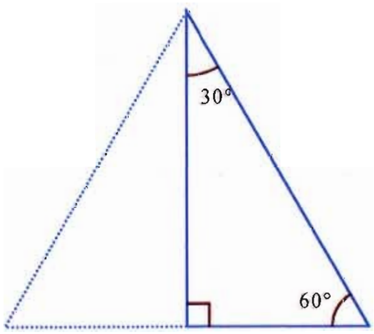
Thus the ratio of the sides of this triangle is $1 : 1 : \sqrt{2}$

- Now another right angled triangle:

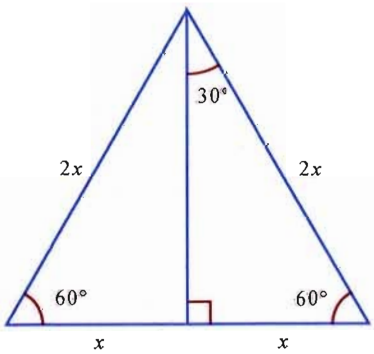


Let's take the length of its shortest side as x. How do we compute the lengths of the other two sides?

The first triangle we saw is half a square; and this one is half an equilateral triangle. (See the section, **Set a square and a triangle,** of the lesson **Between the lines**, in the Class 7 textbook.)



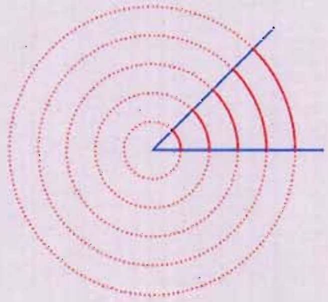
The sides of this equilateral triangles are as shown below:



Degree measure of angles

What does it mean, when we say that an angle is 45°?

We can draw several circles centred at the vertex of such an angle. And the lengths of the arcs of such circles within the sides of this angle are all different.



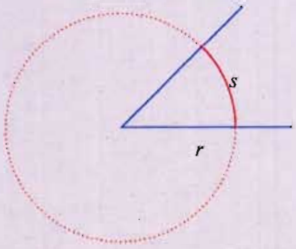
But each of these arcs is $\frac{1}{8}$ of the corresponding full circle. And 45 is the number got by multiplying this fraction $\frac{1}{8}$ by 360.

What if the measure of the angle is 60°? For any circle centred at the vertex of an angle of this size, the length of the arc within the sides of the angle would be $\frac{1}{6}$ of the entire circle. And 60 is the number got by multiplying this $\frac{1}{6}$ by 360.

Generally speaking, the degree measure of any angle is the number got by first drawing a circle centred at its vertex, dividing the length of the arc within its sides by the circumference, and then multiplying this number by 360.

Another measure of an angle

We have seen what the degree measure of an angle means.



We noted that for circles of different sizes, r and s in the picture above change, but $\frac{s}{2\pi r}$ does not change; and the degree measure of the angle is this fixed number multiplied by 360. In other words,

degree measure of the angle $= \frac{s}{2\pi r} \times 360$.

In this, we can change r and s, but not the numbers 2π and 360. So, isn't it enough if we take $\frac{s}{r}$ as a measure of the angle?

That's right. This gives another measure of the angle, called its radian measure. Thus,

radian measure of the angle $= \frac{s}{r}$

We use the symbol ° to denote the degree measure, right? The radian measure is written rad.

This idea was first proposed by the English mathematician, Roger Cotes in the eighteenth century. The name radian was first used by the English physicist, James Thomson in the nineteenth century.

So, we find the length of the hypotenuse of our original triangle as $2x$ and the length of one of the shorter sides to be x. What about the length of the third side?

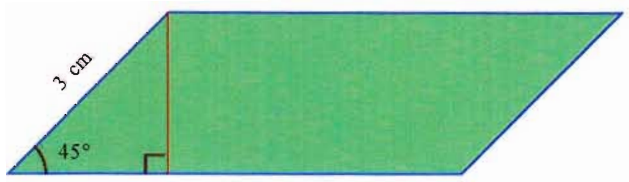
$$\sqrt{(2x)^2 - x^2} = \sqrt{3x^2} = \sqrt{3}x$$

Thus the ratio of the sides of the triangle is $1 : \sqrt{3} : 2$

Let's do some problems using these ideas:

- The adjacent sides of a parallelogram are 6 and 3 centimetres long and the angle between them is $45°$. What is its area?

  To compute the area of a parallelogram, we should know the distance between a pair of parallel sides. Let's draw a figure:

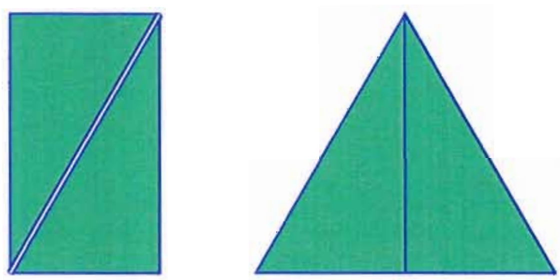


The vertical side of the right angled triangle in this is $\frac{1}{\sqrt{2}}$ of the hypotenuse. (How?) So, the height of the parallelogram is $\frac{3}{\sqrt{2}}$ centimetres. Thus the area of the parallelogram is $\frac{18}{\sqrt{2}}$ square centimetres. If we are willing to compute a bit more, we can write

$$\frac{18}{\sqrt{2}} = 18 \times \frac{\sqrt{2}}{2} \approx 9 \times 1.414 = 12.726$$

This gives the area of the parallelogram as 12.73 square centimetres, correct to two decimal places.

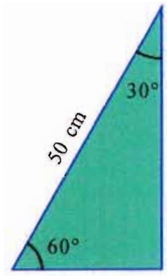
- A rectangular piece of wood is to be cut along the diagonal and the pieces re-arranged to form an equilateral triangle, as shown below and the sides of the triangles should be 50 centimetres long.



What should be the dimensions of the rectangle?

To make an equilateral triangle like this, the triangles got by cutting the rectangle should have angles 30°, 60°, 90°. And the side of the equilateral triangle would be hypotenuse of this right angled triangle.

So, what is our problem? We have to find the lengths of the other two sides of the triangle shown below:



The ratio of the lengths of the sides of this triangle, in the order of size, is $1 : \sqrt{3} : 2$. So, the length of the shortest side is

$$50 \times \frac{1}{2} = 25$$

and the length of the other side is

$$50 \times \frac{\sqrt{3}}{2} = 25\sqrt{3}$$

Thus the lengths of the sides of the rectangle should be 25 centimetres and $25\sqrt{3}$ centimetres. If needed, we can compute these correct to millimetres (Try!)

Some more problems are given below. Try your hand at them.

- The area of a parallelogram is 30 square centimetres. One of its sides is 6 centimetres and one of its angles is 60°. What is the length of its other side?
- The sides of an equilateral triangle are 4 centimetres long. What is the radius of its circumcircle?
- One angle of a right angled triangle is 30° and its hypotenuse is 4 centimetres. What is its area?

Degree and radian

Just as centimetre and inch are two of the several units used to measure length, degree and radian are two common units to measure an angle. In the International System of Units abbreviated as SI units, the Unit of angle measurement is taken as radian.

From the equations defining degree and radian, we see that

radian measure of angle

$$= \text{degree measure of angle} \times \frac{180}{\pi}$$

That is,

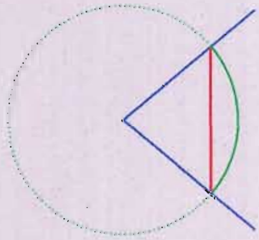
$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^{\circ} \approx 57.2958^{\circ}$$

More easy to remember is the conversion formula

$$\pi \text{ rad} = 180^{\circ}$$

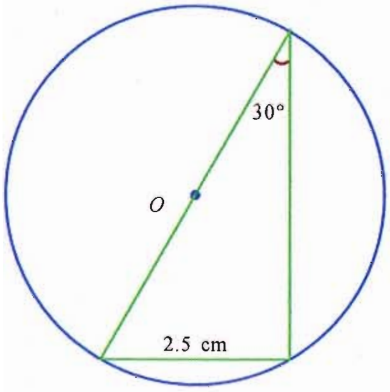
Straightening

In measuring an angle, we actually measure the length of an arc of a circle, whether we use degrees or radians. Instead of this, the Greek astronomer Hipparchus started using lengths of chords, in the second century BC.



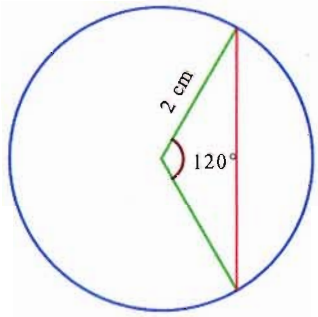
Later mathematicians often refer to a table of chords of various central angles computed by Hipparchus, but this table has not been found. However, such a table of chords done by the Egyptian astronomer Claudius Ptolemy in the second century AD, has been found. He has computed accurately the lengths of chords of central angles up to $180°$ in a circle of radius 60 units, at $\frac{1}{2}^\circ$ intervals.

- In the figure below, O is the centre of the circle.

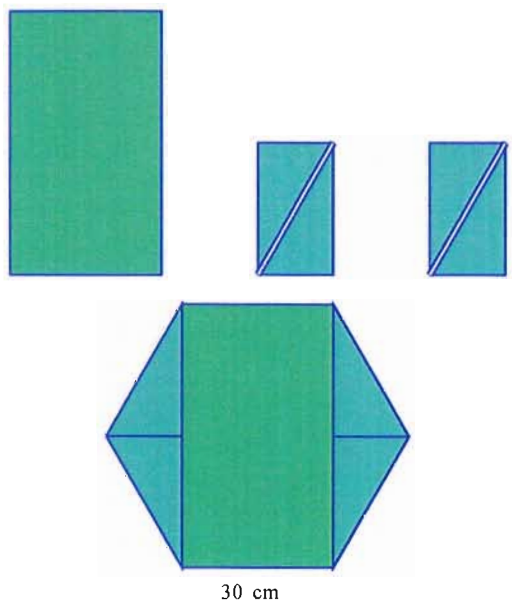


What is the diameter of the circle?

- What is the length of the chord in the figure below?

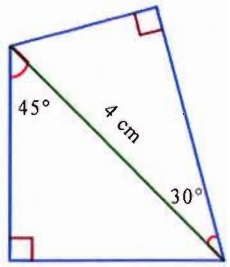


- Two identical rectangles are to be cut along the diagonals and the triangles got joined with another rectangle, to make a regular hexagon as shown below:

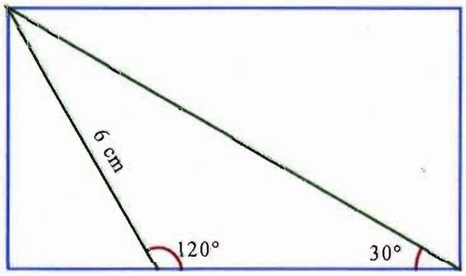


What should be dimensions of the rectangles?

- Compute the lengths of all sides of the quadrilateral below:



- Compute the area of the rectangle below.

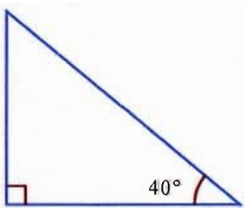


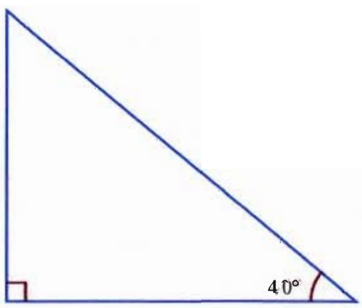
New measures of angles

Recall that our aim is to determine the ratio of the sides of a triangle in terms of its angles. And we have seen how this is done for certain right angled triangles. But this is not so easy for other triangles. Some tables which help us to do this have been computed by mathematicians, quite some time ago. Let's see what these tables are and how we can use them in our task.

First note that for any given angle smaller than a right angle, we can draw any number of right angled triangles having this as one of its angles; and the other angles of all these triangles are also equal.

For example, look at the pictures of some right angled triangles with one angle 40°.

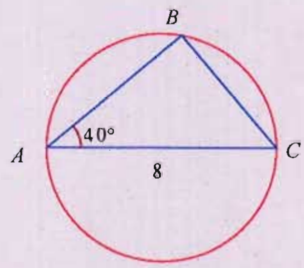




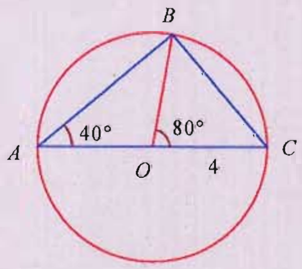
Old methods

Using Ptolemy's tables, we can compute the lengths of the perpendicular sides of a right angled triangle, given its hypotenuse and one angle.

For example, suppose we want to do this for a right angled triangle of hypotenuse 8 centimetres and one angle 40°. What Hipparchus and Ptolemy do is to imagine such a triangle drawn within a circle as shown below:



If we draw the radius to the right angled corner, we get a figure like this:

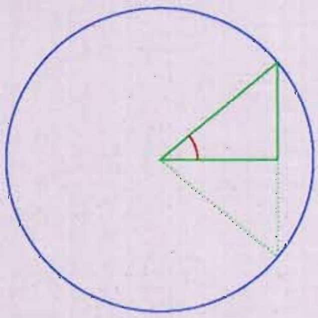


Now using the table of chords, we can find the length of the chord of central angle 80° in a circle of radius 1 unit. Multiplying this by 4 gives one side of our triangle. The other side can be found using Pythagoras Theorem.

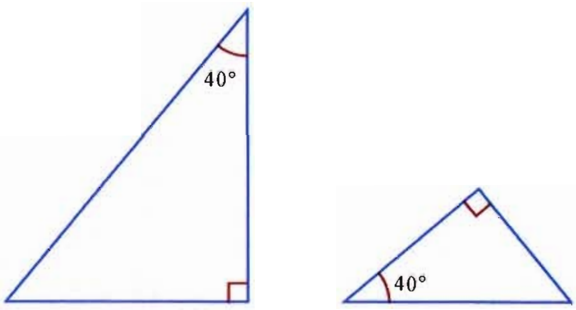
Half-chord

To compute the perpendicular sides of a triangle using Ptolemy's table, we have to double the angle and halve the hypotenuse.

This can be avoided by forming a table which associates with every angle, half the length of the chord of double the angle.



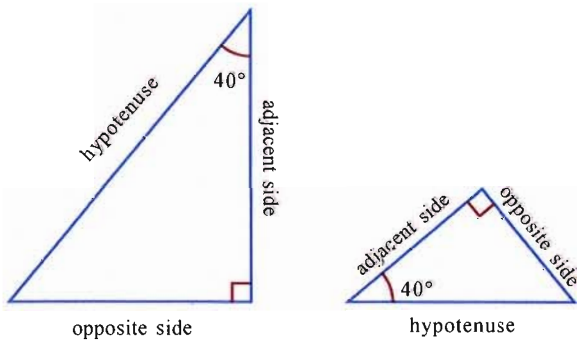
Such a table can be seen in the astronomy text called *Suryasiddhanta*, written in India during the fifth century AD. Operations using such tables can also be found in the book *Aryabhatiya* written by the famous Indian astronomer Aryabhata in the same period. He calls this measure of the angle *ardhajya.* (We have noted in the section **Chord and cord,** of the lesson **Circle** in the Class 9 textbook, that that chord of a circle is called *jya* in Sanskrit.)



Though these are of different sizes, all of them have the same three angles 40°, 50°, 90°. And so the lengths of their sides are in the same ratio.

In other words, the ratio of the lengths of two sides of any one triangle among these, is equal to the ratio of the lengths of the sides in the same position of any other triangle.

To shorten this some what, let's call the shorter of the two sides containing the angle 40°, its *adjacent side*. The longer side is of course, the hypotenuse. The side opposite this angle, naturally enough, we name its *opposite side.*



Then in each of these triangles, the number got by dividing the opposite side of 40° by the hypotenuse is the same; and this has been computed to be about 0.6428. Again, the number got by dividing the adjacent side of 40° by the hypotenuse is also the same for all triangles and this has been computed to be about 0.7660.

These numbers have special names. For example, the number got by dividing the opposite side of 40° by the hypotenuse, in any right angled triangle with this as an angle, is called the *sine* of 40°; and the number got by dividing the adjacent side of 40° by the hypotenuse is called the *cosine* of 40°.

They are shortened as sin 40° and cos 40°.

Thus as mentioned earlier,

$$\sin 40° \approx 0.6428$$

$$\cos 40° \approx 0.7660$$

There are tables which give the sin and cos for all angles less than 90°. A part of it looks like this (The full table is given at the end of the chapter.)

Angle	sin	cos
35°	0.5736	0.8192
36°	0.5878	0.8090
37°	0.6018	0.7986
38°	0.6157	0.7880
39°	0.6293	0.7771
40°	0.6428	0.7660

From this we see for example,

$$\sin 35° \approx 0.5736$$

$$\cos 35° \approx 0.8192$$

What do these mean? In any right angled triangle drawn with one angle 35°, the opposite side of this angle divided by the hypotenuse gives approximately 0.5736; and its adjacent side divided by the hypotenuse gives approximately 0.8192.

Using these names, we can describe the facts about the right angled triangles seen earlier as

$$\sin 45° = \frac{1}{\sqrt{2}} \qquad \cos 45° = \frac{1}{\sqrt{2}}$$

$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2}$$

Can you write sin 30° and cos 30° like this?

Now let's look at some instances of using these tables:

- The hypotenuse of a right angled triangle is 6 centimetres long and one of its angles is 40°. What are the lengths of its other two sides?

What's in a name?

The measure of an angle that is now called sin is the same as what Aryabhata called *ardhajya*. This is how the name evolved.

Arybhata himself, in his later works, drops the adjective *ardha* (meaning half) and writes only *jya* for the half-chord associated with an angle. For a period starting sometime around seventh century AD, the rulers of the Arab countries actively promoted the translation of ancient texts from Greece and India. In the translation of *Aryabhatiya*, the word *jya* was transliterated as *jiba*. Following the custom of not writing the vowels, this was written as *jb* in Arabic.

Later, these Arabic texts reached Europe and were translated into Latin, sometime around the thirteenth century. In supplying the missing vowels to *jb*, the Latin translators mistook it for the word *jaib* which means a bend or a fold. They used the word *sinus* which means the same thing in Latin. During the course of time this became simply sine.

The word cosine comes from what Arybhata calls the *kotijya*.

Kerala math

We have mentioned Madhavan, the fourteenth century Kerala mathematician. (the section **π in Keralam** of the lesson **Circle Measures** in the Class 9 textbook). He discovered a sequence to compute the length of a chord from the length of its arc. Translating what he has written in Sanskrit to modern mathematical notation, his finding is that for any number x, the sequence:

$$x,$$

$$x - \frac{x^3}{1\times2\times3},$$

$$x - \frac{x^3}{1\times2\times3} + \frac{x^5}{1\times2\times3\times4\times5}$$

....

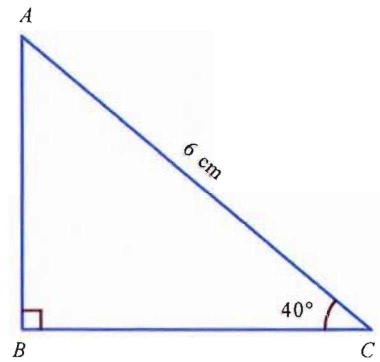
gets closer and closer to $\sin x$, where x is in radians

We can shorten this by writing

$$\sin x = x - \frac{x^3}{1\times2\times3} + \frac{x^5}{1\times2\times3\times4\times5} - \cdots$$

This was re-discovered by Newton in England and Leibniz in Germany, during the seventeenth century.

Let's draw a picture:



From this figure, we get

$$\frac{AB}{AC} = \sin 40° \text{ and } \frac{BC}{AC} = \cos 40°$$

and these give

$$AB = AC \times \sin 40°$$

$$BC = AC \times \cos 40°$$

Now using the given fact that $AC = 6$ centimetres and the values of $\sin 40°$ and $\cos 40°$ got from the tables, we can compute

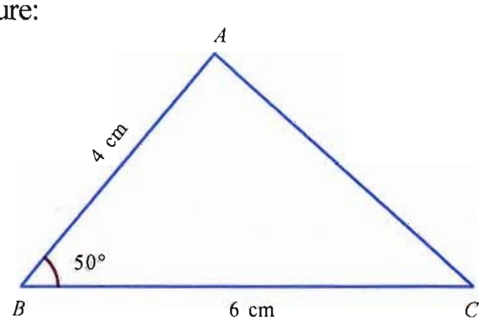
$$AB \approx 6 \times 0.6428 = 3.8568$$

$$BC \approx 6 \times 0.7660 = 4.596$$

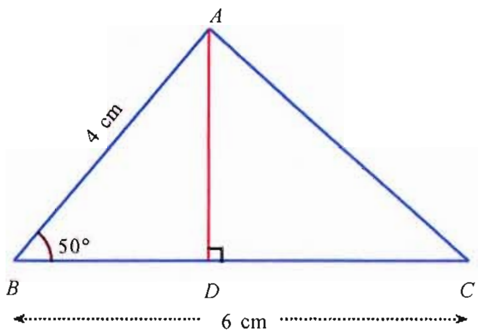
Thus the perpendicular sides of the triangle are approximately 3.9 and 4.6 centimetres long.

- The lengths of two sides of a triangle are 6 centimetres and 4 centimetres; and the angle between them is 50°. What is the area of this triangle?

See this figure:



To compute the area of this triangle, we need the height from a side. Let's draw the perpendicular from the top vertex.



The area of the triangle is

$$\frac{1}{2} \times BC \times AD = \frac{1}{2} \times 6 \times AD = 3 \times AD$$

How can we compute the length of AD? From the right angled triangle ABD in the figure, we get

$$\frac{AD}{AB} = \sin 50°$$

From this we get

$$AD = AB \times \sin 50° = 4 \sin 50°$$

And from the tables we find

$$\sin 50° \approx 0.7660$$

so that we get

$$AD \approx 4 \times 0.7660 = 3.064$$

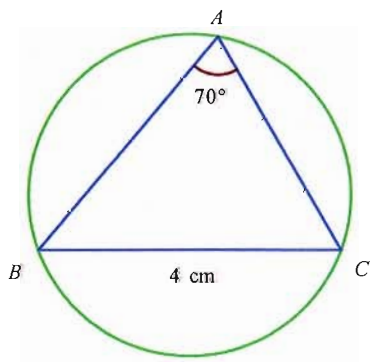
Now we can find the area as

$$3 \times AD \approx 3 \times 3.064 \approx 9.19$$

Thus the area of the triangle is about 9.19 square centimetres.

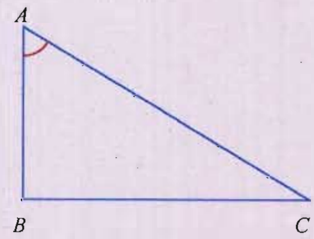
Now take the angle at B as 130° instead of 50°, and compute the area.

- One angle of a triangle is 70° and the length of its opposite side is 4 centimetres. What is its circumradius?



Pythagorean relation

See this triangle:



The Pythagoras Theorem applied to it gives

$$AB^2 + BC^2 = AC^2$$

Dividing this equation by AC^2, we get

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = 1$$

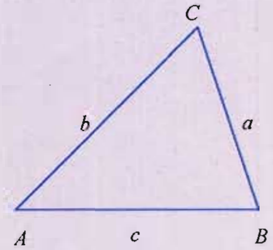
Viewing this relative to $\angle A$, it becomes

$$\cos^2 A + \sin^2 A = 1$$

And this is true for any angle. (The squares of $\cos A$ and $\sin A$ are written $\cos^2 A$ and $\sin^2 A$.)

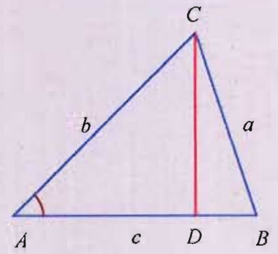
Area

Look at the traingle below.



How do we compute the area of this triangle?

For this, we first draw the perpendicular from C.



Then we have

$$\text{area} = \frac{1}{2} \times AB \times CD$$

Also, from the figure, we see that

$$CD = AC \sin A = b \sin A$$

So, we get

$$\text{area} = \frac{1}{2}\, bc \sin A$$

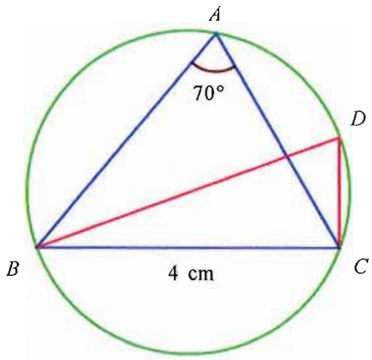
Drawing perpendiculars from other vertexes, we can also see that

$$\text{area} = \frac{1}{2}\, ca \sin B = \frac{1}{2}\, ab \sin C$$

All these expressions for area give the same number, right? Can you find any relation between the sides and angles of a triangle from these?

Have we done recently any such problem, where we had to compute a diameter? Just have another look at the problems done so far in this lesson.

Let's draw the diameter through B in the above figure and join its other end with C:



Then BCD is a right angled triangle (why?) and also, the angles at A and D are equal (reason?) So, $\angle BDC = 70°$. Now from the right angled triangle BDC, we get

$$\frac{BC}{BD} = \sin 70°$$

and from this

$$BD = \frac{BC}{\sin 70°} = \frac{4}{\sin 70°}$$

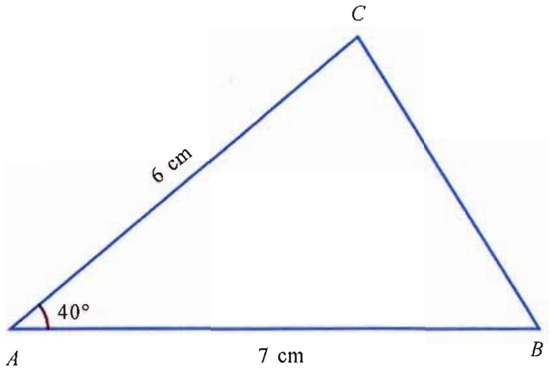
From the tables, we can see that $\sin 70° \approx 0.9397$ and then we can compute

$$BD = \frac{4}{0.9397} \approx 4.3$$

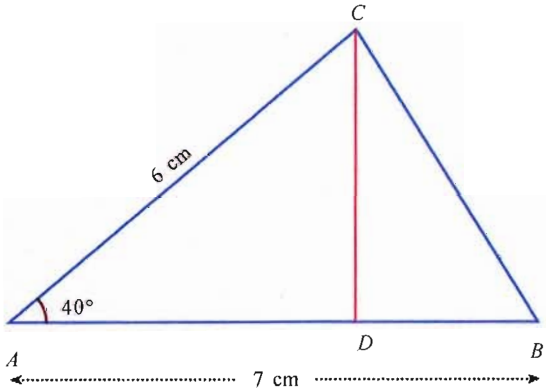
using a calculator. Thus the diameter of the cicumcircle is about 4.3 centimetres.

Can you compute the circum radius if the angle is 110° instead of 70° in this problem?

- Two sides of a triangle are 7 and 6 centimetres long and the angle between them is 40°. What is the length of the third side?



To find the length of BC, the trick is to draw the perpendicular from C to AB.



Now from the right angled triangle BCD, we get

$$BC^2 = BD^2 + DC^2$$

Now let's see how we can compute BD and DC.

From the right angled triangle ACD, we get

$$DC = AC \sin 40° \approx 6 \times 0.6428 \approx 3.86$$

Again from the same triangle,

$$AD = AC \cos 40° \approx 6 \times 0.7660 \approx 4.60$$

This gives

$$BD = AB - AD \approx 7 - 4.6 = 2.4$$

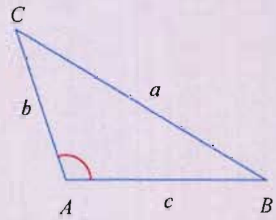
Now we can compute

$$BC = \sqrt{BD^2 + DC^2} \approx \sqrt{3.86^2 + 2.4^2} = 4.54$$

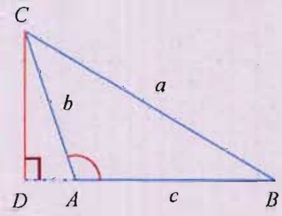
Thus the length of BC is about 4.5 centimetres. What would be the length of BC, if the angle at A is 110° instead?

Large angles

What is the area of this triangle?



If we draw the perpendicular from C, we get the figure below:



Here also,

$$\text{area} = \frac{1}{2} \times AB \times CD$$

But we cannot write CD as $b \sin A$ (why not?)

However, from the right angled triangle ADC, we have

$$\angle CAD = 180° - \angle CAB$$

so that

$$CD = b \sin (180° - \angle CAB)$$

Now let's write $\angle A$ instead of $\angle CAB$ (this is anyway the interior angle at A). Then we can write

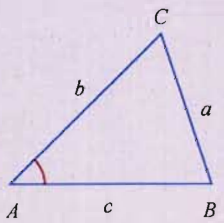
$$\text{area} = \frac{1}{2} bc \sin (180 - A)$$

In general, if $\angle A < 90°$ in ΔABC, then its area is $\frac{1}{2} bc \sin A$; if $\angle A > 90°$, then the area is $\frac{1}{2} bc \sin (180 - A)$.

What if $\angle A = 90°$?

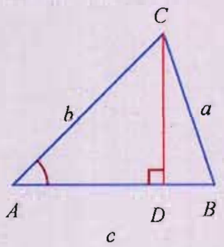
The third side

See this triangle:



Suppose we know the lengths b, c and the angle A. How do we find the length a?

For this also, we need only draw the perpendicular from C.



From the right angled triangle ADC, we get

$$AD = b \cos A \text{ and } CD = b \sin A$$

Now from the right angled triangle BDC, we get

$$a^2 = b^2 \sin^2 A + (c - b \cos A)^2$$

In this, if we write

$$(c - b \cos A)^2 = c^2 + b^2 \cos^2 A - 2bc \cos A$$

and also see that,

$$b^2 \sin^2 A + b^2 \cos^2 A = b^2 (\sin^2 A + \cos^2 A) = b^2$$

then we get

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Now some problems for you:

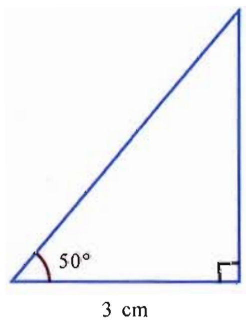
- Without actually drawing figures or looking up tables,can you arrange the numbers below in ascending order?

$$\sin 1°, \cos 1°, \sin 2°, \cos 2°$$

- The lengths of two sides of a triangle are 6 centimetres and 4 centimetres; and the angle between them is 130°. What is its area?
- One angle of a triangle is 110° and the side opposite to it is 4 centimetres long. What is its circumradius?
- Two sides of a triangle are 7 and 6 centimetres long and the angle between them is 140°. What is the length of the third side?
- Two sides of a parallelogram are of length 6 centimetres and 4 centimetres and the angle between them is 35°. What are the lengths of its diagonals?

Another measure

We want to draw a right angled triangle with one of the shorter sides 3 centimetres long and an angle on it 50°.

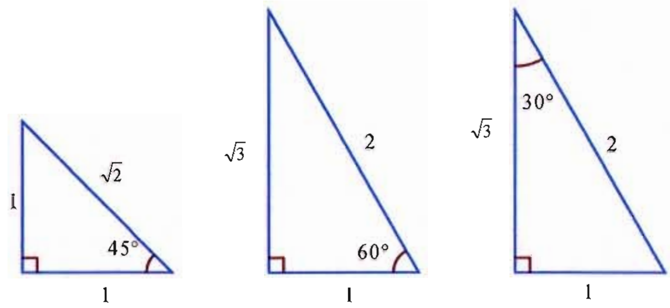


Not a difficult job, right? What is the length of the other short side?

If we look up cos 50° in the table, then we can compute the hypotenuse and then the third side using Pythagoras Theorem.

We can use another table to compute this directly. The numbers got by dividing the opposite side by the adjacent side of an angle in various right angled triangles have also been tabulated.

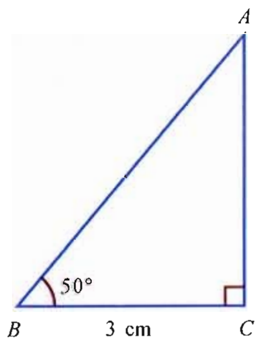
The number thus got is called the *tangent* of the angle and is shortened as tan. As examples, let's look at some of the triangles seen earlier.



We can then see that

$\tan 45° = 1$ $\qquad$ $\tan 60° = \sqrt{3}$ $\qquad$ $\tan 30° = \frac{1}{\sqrt{3}}$

Let's return to our original problem:



As mentioned just now, we can see that

$$\frac{AC}{BC} = \tan 50°$$

and then compute

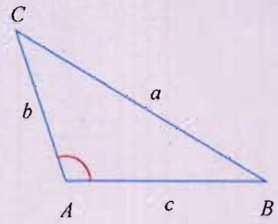
$$AC = BC \times \tan 50° \approx 3 \times 1.1918 = 3.5754 \approx 3.6$$

using the figure and the tables. Thus the required length of the side is about 3.6 centimetres.

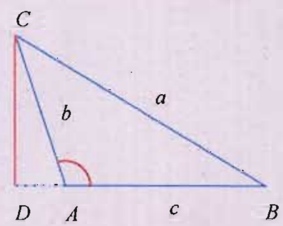
Here's another instance of using the tan measure.

We want to find out how high the man in the picture stands above the ground.

Large angles again



Suppose in the above figure, we know the lengths b, c and the angle A. Does the earlier method of computing the length a of the third side work for this also?



What all things are changed?

From the right angled triangle ADC, what we get here are

$$AD = b \cos\ (180 - A)$$

$$CD = b \sin\ (180 - A)$$

Also, what we get from the right angled triangle BDC becomes

$$BD = c + b \cos (180 - A)$$

With these changes, our earlier technique gives here

$$a^2 = b^2 + c^2 + 2bc \cos (180 - A)$$

What we saw here is a triangle with $\angle A > 90°$. What if $\angle A = 90°$?

New meanings

To compute the sin, cos or tan of an angle, we have to first draw a right angled triangle with this as one of its angles. This is possible only if the angle at hand is less than 90°. Thus so far as we have seen, trigonometric measures are possible only for angles less than 90°.

We have seen that because of this limitation, we need different formulas depending on the size of the angle in various computations, such as finding the area of a triangle or the lengths of its sides. To overcome this, we make new definitions of sin and cos for angles greater than 90°. They are as follows:

$$\sin(180° - x) = \sin x$$
$$\cos(180° - x) = -\cos x$$

We extend the definitions also as

$$\sin 90° = 1$$
$$\cos 90° = 0$$

(Visualize what happens to the opposite side and adjacent side of an angle in a right angled triangle, when it gets closer and closer to 90°)

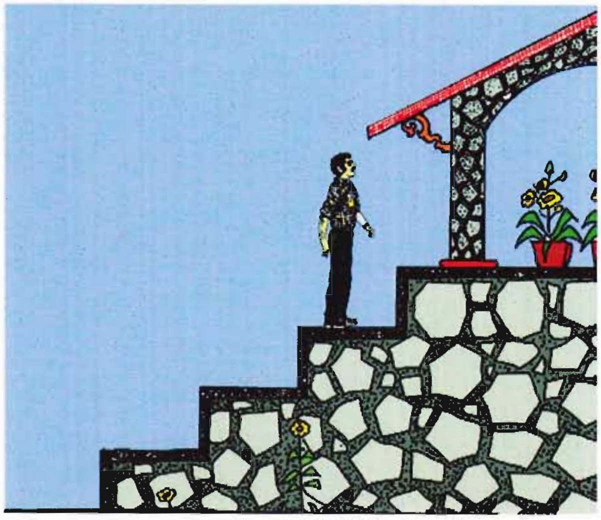
With these new definitions, we can use the single formula

$$\text{area} = \frac{1}{2} bc \sin A$$

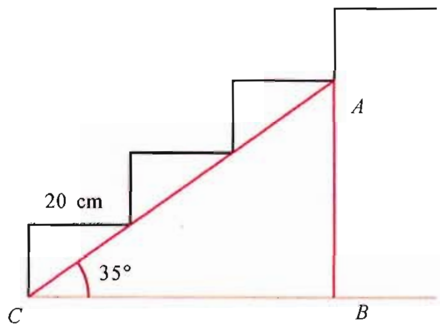
and

$$a^2 = b^2 + c^2 - 2bc \cos A$$

in any triangle with angles A, B, C and sides a, b, c.



The dimensions of the steps are as shown below:



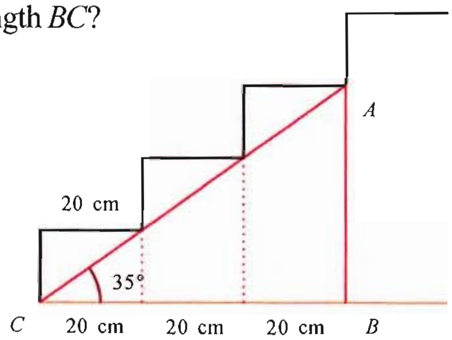
What we are asked to compute is the height AB. From the above figure, we get

$$AB = BC \times \tan 35°$$

and from the tables we get

$$\tan 35° \approx 0.7002$$

What about the length BC?



From the above figure, we can see that the length of BC is 60 centimetres. Thus

$$AB = BC \times \tan 35° \approx 60 \times 0.7002 = 42.012$$

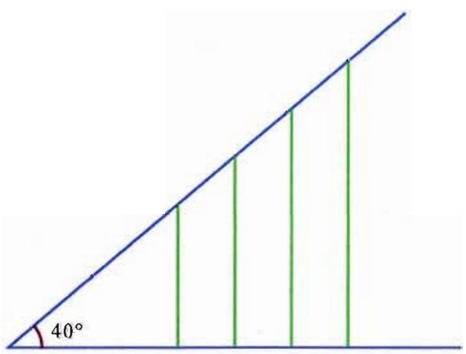
So, the height is about 42 centimetres.

Now some problems for you to do on your own:

- How many rhombuses can we draw with one diagonal 5 centimetres long and one angle 50° ? What are their areas?
- A ladder leans against a wall with its foot 2 metres away from the wall and it makes a 40° angle with the ground. How high is the top of the ladder from the ground?
- Three rectangles are cut along their diagonals and the triangles so got are rearranged to form a regular pentagon as shown below:

Find the dimensions of the rectangles.

- The vertical lines in the figure below are drawn 1 centimetre apart.

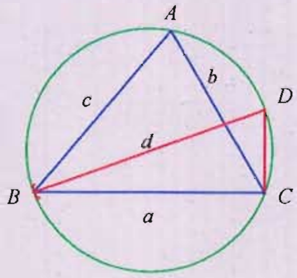


Prove that their heights are in arithmetic sequence.

What is the common difference?

Triangles and circles

See this picture:



BD is a diameter of the circle. So, $\angle BCD = 90°$ and $\angle D = \angle A$. Taking the diameter of the circle as d, we get

$$a = d \sin D = d \sin A$$

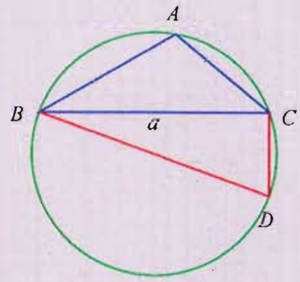
from the right angled triangle BCD. Similarly, we can see that

$$b = d \sin B,\ c = d \sin C.$$

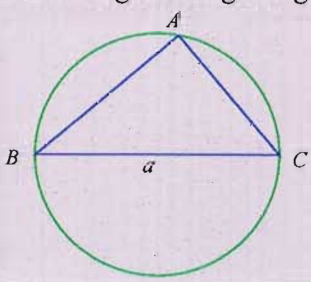
So,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = d$$

Would this be right if an angle of the triangle is larger than 90°?



What if one angle is a right angle?



Problem solved!

We saw that in any triangle (whatever type it is) with angles A, B, C and the lengths of sides opposite them a, b, c, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

In other words,

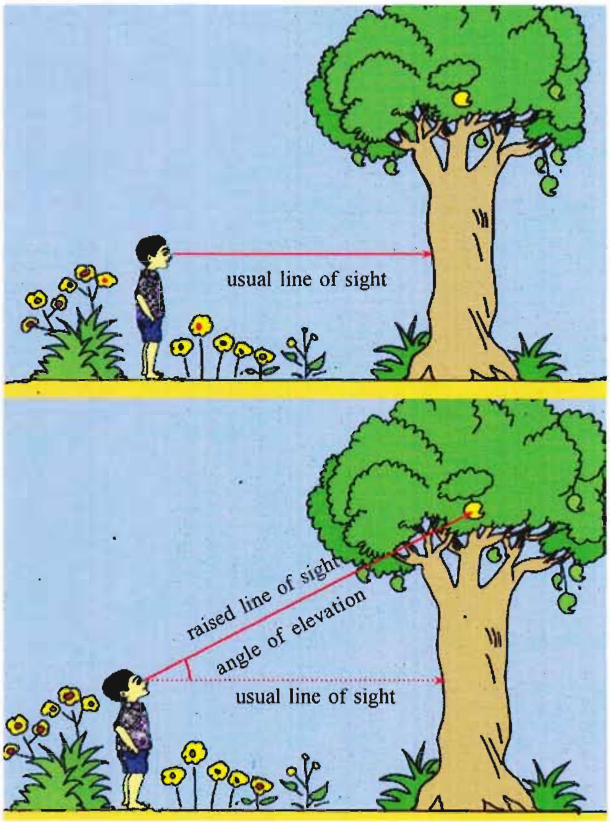
$$a : b : c = \sin A : \sin B : \sin C$$

That is, in any triangle, the ratio of sides is equal to the ratio of the sines of the angles opposite them.

We started by observing that in all the various triangles having the same three angles, the ratio of the lengths of sides is the same; and then set out to find this unchanging ratio. Now we have the answer.

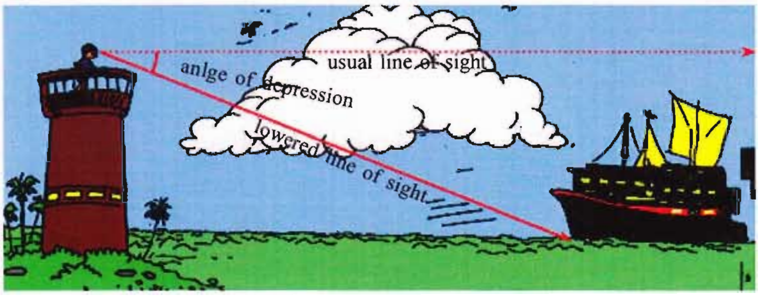
Heights and distances

To see things high up, we have to raise our heads. See these pictures



Our usual line of sight is parallel to the ground and when we look up at something high, this is raised. The angle between these lines is called the *angle of elevation.*

Similarly when we look down from a height, we have to lower our line of vision.



The angle so formed is called the *angle of depression.*

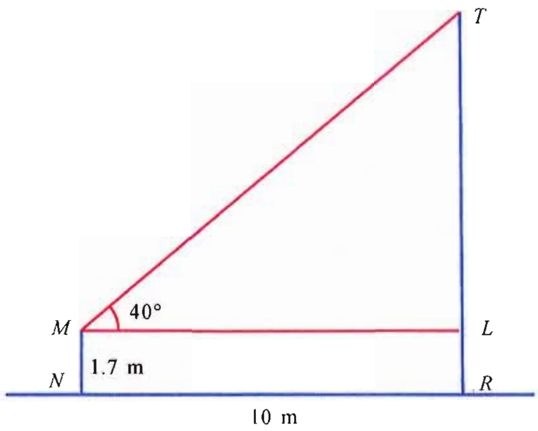
Such angles are measured using an instrument called *clinometer*.

Heights and distances which cannot be directly measured are computed by measuring angles using a clinometer and doing calculations using sin and cos.

Let's look at some examples:

- A man 1.7 metres tall stands 10 metres away from the foot of a tree; and he sees the top of the tree at an angle of elevation 40°. How tall is the tree?

  In the figure below, the line MN denotes the man and TR, the tree.



From the figure and with the help of tables, we get

$$TL = ML \tan 40° \approx 10 \times 0.8390 = 8.39$$

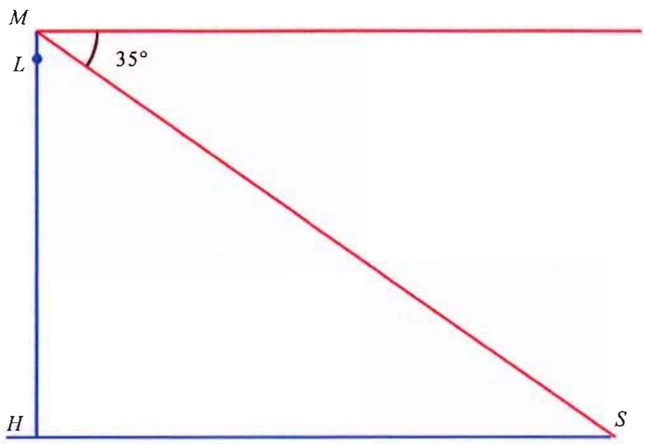
So,

$$TR = TL + LR = TL + MN \approx 8.39 + 1.7 = 10.09$$

Thus the height of the tree is about 10.09 metres.

- A man 1.8 metres tall looks down from the top of a lighthouse 25 metres high and sees a ship at an angle of depression 35°. How far is the ship from the foot of the lighthouse?

  Let's draw a figure first:



In this, LH is the lighthouse and LM is the man standing on top. The point S denotes the ship. Thus we want to find the length HS.

Congruency

We have seen in the lesson **Congruent Triangles** of the Class 8 textbook, that in a triangle if any of the three measures, (i) the lengths of all three sides, (ii) the lengths of two sides and the angle between them or (iii) the length of one side and the two angles on it, are specified, then all the other measures are fixed.

How do we compute them?

Suppose we know the lengths a, b, c of a triangle. Then the angles A, B, C can be computed using relations like

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

If we know the lengths a, b of two sides and the included angle C, then we can first find the length c of the remaining side using the equation

$$c^2 = a^2 + b^2 - 2ab \cos C$$

and then compute the remaining angles as in the first case.

If we know the length a of one side and the two angles B and C on it, then we first find the third angle A by

$$A = 180 - (B + C)$$

and then find b and c using the equations

$$b = \frac{a \sin B}{\sin A}, \quad c = \frac{a \sin C}{\sin A}$$

Thus we complete the determination of triangles using trigonometry.

Slant and spread

We have noted that the trigonometric measures sin and cos arose out of the need to see angle as spread. The measure tan is the result of connecting this method with the need to consider angle as a slant. (Its definition is nothing but the old way of measuring slant as the quotient of rise by run.)

Such a connection was first made by the ninth century Arab mathematician Ahmed ibn Abdallah al Mervazi. He also gave a table of tan measures.

The name tangent for this measure originated only in the sixteenth century. (see the section, **Name and meaning** of the lesson **Tangents**)

Using the facts given in the problem, we can find

$$MH = ML + LH = 25 + 1.8 = 26.8$$

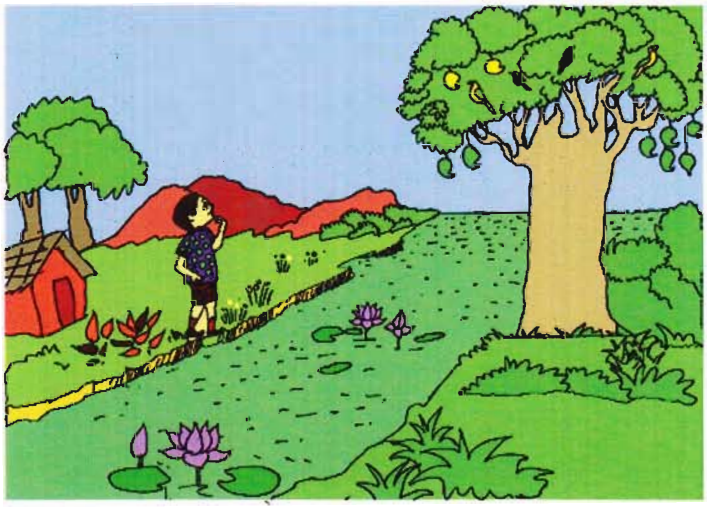
and $\angle HMS = 55°$

So, from the right angled triangle MHS, we get

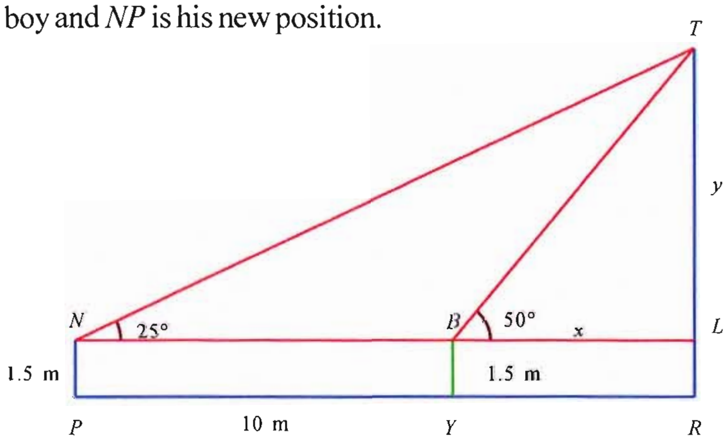
$$HS = MH \tan 55° \approx 26.8 \times 1.4281 \approx 38.27$$

This means the ship is about 38.27 metres away from the foot of the light house.

- A boy, 1.5 metres tall, standing at the edge of a river bank, sees the top of a tree on the the edge of the other bank at an elevation of 50°. Standing back by 10 metres, he sees it at an elevation of 25°. How wide is the river and how tall is the tree?



In the figure below, TR is the tree, BY is the first position of the boy and NP is his new position.



What we want to compute are the lengths of YR and TR.

From the figure, we get

$YR = BL$ and $TR = TL + LR = TL + 1.5$

So, we need only find BL and TL. If we take $BL = x$ and $TL = y$, then from the right angled triangle BTL, we get

$$y = x \tan 50° \approx 1.1918x$$

and from the right angled triangle NTL,

$$y = (x + 10) \tan 25° \approx 0.4663\ (x + 10) = 0.4663x + 4.663$$

These equations show that

$$1.1918x \approx 0.4663x + 4.663$$

from which we can find

$$x \approx \frac{4.663}{0.7255} \approx 6.427$$

using a calculator. Then we can find

$$y \approx 1.1918 \times 6.427 \approx 7.659$$

Thus the width of the river is about 6.43 metres and the height of the tree is about $7.66 + 1.5 = 9.16$ metres.

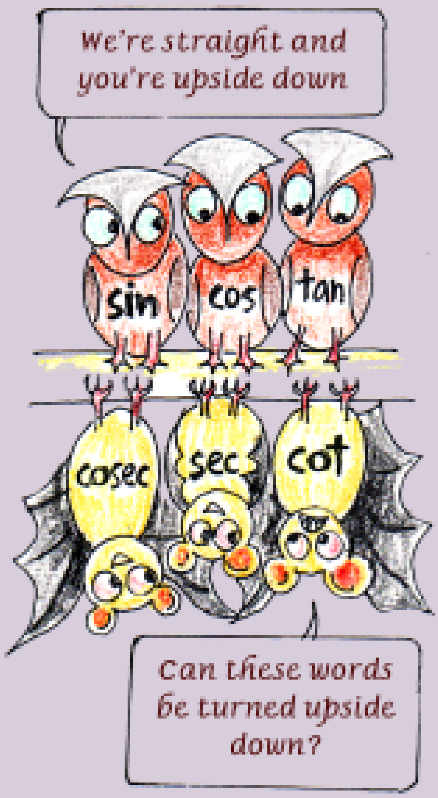
Now you can try these problems:

- The length of the shadow of a tree is 18 metres, when the sun is at an elevation of 40°. What is the height of the tree?
- A man 1.75 metres tall, standing at the foot of a tower sees the top of a hill 40 metres away at an elevation of 60°. On climbing to the top of the tower, he sees the top of the hill at an elevation of 50°. Compute the heights of the hill and the tower.
- A boy 1.5 metres tall, sees the top of a building under construction at an elevation of 30°. The building is completed, adding 10 more metres to its height; and then the boy sees the top at an elevation of 60° from the same spot. What is the total height of the completed building?
- A man 1.8 metres tall, looking down from the top of a telephone tower sees the top of a building 10 metres high at an angle of depression 40° and the foot of the building at an angle of depression 60°. What is the height of the tower? How far is it away from the building?

Other measures

We saw how measures such as sin, cos and tan are defined, by drawing a right angled triangle including the angle and then forming the quotients of the lengths of its sides in various ways. There are some more quotients remaining, which we have not so far considered. These too have special names.

The reciprocals of the sin, cos and tan of an angle are called its *cosecant*, *secant* and *cotangent*; and they are abbreviated as cosec, sec and cot.



Trigonometric Tables

Angle	sin	cos	tan	Angle	sin	cos	tan
0°	0.0000	1.0000	0.0000	**46°**	0.7193	0.6947	1.0355
1°	0.0175	0.9998	0.0175	**47°**	0.7314	0.6820	1.0724
2°	0.0349	0.9994	0.0349	**48°**	0.7431	0.6891	1.1106
3°	0.0523	0.9986	0.0524	**49°**	0.7547	0.6561	1.1504
4°	0.0698	0.9976	0.0699	**50°**	0.7660	0.6428	1.1918
5°	0.0872	0.9962	0.0875	**51°**	0.7771	0.6293	1.2349
6°	0.1045	0.9945	0.1051	**52°**	0.7880	0.6157	1.2799
7°	0.1219	0.9925	0.1228	**53°**	0.7986	0.6018	1.3270
8°	0.1392	0.9903	0.1405	**54°**	0.8090	0.5878	1.3764
9°	0.1564	0.9877	0.1584	**55°**	0.8192	0.5736	1.4281
10°	0.1736	0.9848	0.1763	**56°**	0.8290	0.5592	1.4826
11°	0.1908	0.9816	0.1944	**57°**	0.8387	0.5446	1.5399
12°	0.2079	0.9781	0.2126	**58°**	0.8480	0.5299	1.6003
13°	0.2250	0.9744	0.2309	**59°**	0.8572	0.5150	1.6643
14°	0.2419	0.9703	0.2493	**60°**	0.8660	0.5000	1.7321
15°	0.2588	0.9659	0.2679	**61°**	0.8746	0.4848	1.8040
16°	0.2756	0.9613	0.2867	**62°**	0.8829	0.4695	1.8807
17°	0.2924	0.9563	0.3057	**63°**	0.8910	0.4540	1.9626
18°	0.3090	0.9511	0.3249	**64°**	0.8988	0.4384	2.0503
19°	0.3256	0.9455	0.3443	**65°**	0.9063	0.4226	2.1445
20°	0.3420	0.9397	0.3640	**66°**	0.9135	0.4067	2.2460
21°	0.3584	0.9336	0.3839	**67°**	0.9205	0.3907	2.3559
22°	0.3746	0.9272	0.4040	**68°**	0.9272	0.3746	2.4751
23°	0.3907	0.9205	0.4245	**69°**	0.9336	0.3584	2.6051
24°	0.4067	0.9135	0.4452	**70°**	0.9397	0.3420	2.7475
25°	0.4226	0.9063	0.4663	**71°**	0.9455	0.3256	2.9042
26°	0.4384	0.8988	0.4877	**72°**	0.9511	0.3090	3.0777
27°	0.4540	0.8910	0.5095	**73°**	0.9563	0.2924	3.2709
28°	0.4695	0.8829	0.5317	**74°**	0.9613	0.2756	3.4874
29°	0.4848	0.8746	0.5543	**75°**	0.9659	0.2588	3.7321
30°	0.5000	0.8660	0.5774	**76°**	0.9703	0.2419	4.0108
31°	0.5150	0.8572	0.6009	**77°**	0.9744	0.2250	4.3315
32°	0.5299	0.8480	0.6249	**78°**	0.9781	0.2079	4.7046
33°	0.5446	0.8387	0.6494	**79°**	0.9816	0.1908	5.1446
34°	0.5592	0.8290	0.6745	**80°**	0.9848	0.1736	5.6713
35°	0.5736	0.8192	0.7002	**81°**	0.9877	0.1564	6.3138
36°	0.5878	0.8090	0.7265	**82°**	0.9903	0.1392	7.1154
37°	0.6018	0.7986	0.7536	**83°**	0.9925	0.1219	8.1443
38°	0.6157	0.7880	0.7813	**84°**	0.9945	0.1045	9.5144
39°	0.6293	0.7771	0.8098	**85°**	0.9962	0.0872	11.4301
40°	0.6428	0.7660	0.8391	**86°**	0.9976	0.0698	14.3007
41°	0.6561	0.7547	0.8693	**87°**	0.9986	0.0523	19.0811
42°	0.6691	0.7431	0.9004	**88°**	0.9994	0.0349	28.6363
43°	0.6820	0.7314	0.9325	**89°**	0.9998	0.0175	57.2900
44°	0.6947	0.7193	0.9657	**90°**	1.0000	0.0000	....
45°	0.7071	0.7071	1.0000				