3 Second Degree Equations

New Equations

Let's start with a problem:

- When the length of each side of a square was increased by 5 centimetres, its perimeter became 36 centimetres. What was the length of a side of the original square?

We have seen very many problems like this. How do we solve it?

We can think like this: the length of a side of the new square is $36 \div 4 = 9$, and so the length of a side of the original square is $9 - 5 = 4$.

Or, we can forget about squares and perimeters and reformulate the problem solely in terms of numbers like this.

- The sum of a number and 5, multiplied by 4 gives 36. What is the number?

Thinking in reverse and inverting the operations, we can find the sum of the number and 5 as $36 \div 4 = 9$ and the number itself as $9 - 5 = 4$.

Or again, we may go all the way and formulate the problem in algebra, as:

- Find x such that $4(x + 5) = 36$

And we can solve it by first writing

$$x + 5 = \frac{36}{4} = 9$$

and then

$$x = 9 - 5 = 4.$$

Quantities and equations

We use numbers to denote various quantities and use algebra for expressing unchanging relations between varying quantities. For example, the relation between the length of a side and perimeter of a square can be written as

$$p = 4s$$

Here the unchanging relation is that, whatever be the length of a side, the perimeter is four times this length.

If an object is thrown upwards from the earth with a speed of u metres per second, then its speed after t seconds is given by

$$v = u - 9.8t$$

We can use such equations to find some quantities when other quantities are known. For example, if an object is thrown upwards at the speed of 20m/s, then to find when its speed would be 10m/s, we need only find the number t for which

$$20 - 9.8t = 10$$

Equations and equations

We have seen problems involving just one unknown quantity and the resulting equations in Class 8. All of them, after simplifications, were reduced to equations like $2x = 3$ or $\frac{1}{2}x = -7$.

In general, these equations were all of the form $ax = b$.

But in many instances, we get equations involving the squares of quantities. For example, consider the problem of finding the lengths of the sides of a rectangle of area 323, with one side 2 centimetres longer than the other. To find the answer to this problem, we have to find x satisfying the equation

$$x(x + 2) = 323$$

which in turn becomes

$$x^2 + 2x = 323$$

Let's slightly change our original problem:

- When the length of each side of a square was increased by 5 centimetres, its area became 36 square centimetres. What was the length of a side of the original square?

What is the length of a side of the new square?

How did you get it as 6 centimetres?

So, the length of a side of the original square is $6 - 5 = 1$ centimetre.

How about changing this to a problem involving numbers alone?

- The square of the sum of a number and 5 is 36. What is the number?

To get back the number from its square, we need only take the square root. (In other words, the inverse operation of squaring is extracting the square root.)

Thus the sum of the number and 5 is $\sqrt{36} = 6$

And the number itself is $6 - 5 = 1$

Now how about writing the problem in algebra?

- Find x such that $(x + 5)^2 = 36$

And the answer?

$$x + 5 = \sqrt{36} = 6$$

$$x = 6 - 5 = 1$$

Let's look at another problem:

- The common difference of an arithmetic sequence is 5 and the square of the second term is 36. What is the first term?

Reasoning as in the last problem, we get the second term of the sequence as 6. So, the first term is 1.

In other words, the arithmetic sequence is 1, 6, 11, ...

Is there any other arithmetic sequence with the same property as given in this problem? How about the arithmetic sequence −11, −6, −1, ...?

Why did we miss this? When we reasoned the square of the second term is 36 and so the second term is 6, we forgot the fact that the square of −6 is also 36, right?

So, what is the correct reasoning?

The second term is either 6 or −6. If the second term is 6, then the first term is $6 - 5 = 1$, and if the second term is −6, then the first term is $-6 - 5 = -11$.

What about algebra?

$$(x + 5)^2 = 36$$

$$x + 5 = 6 \text{ or } x + 5 = -6$$

$$x = 6 - 5 = 1 \text{ or } x = -6 - 5 = -11$$

To shorten this a little, we use a symbol. Instead of saying 6 or −6, we write ± 6 (to be read, "plus of minus 6"). Using this, we can write the steps above like this:

$$(x + 5)^2 = 36$$

$$x + 5 = \pm 6$$

$$x = -5 \pm 6$$

$$x = -5 + 6 = 1 \text{ or } x = -5 - 6 = -11$$

But, shouldn't we have thought along these lines in the rectangle problem also?

Do we have to? The length of a side of a rectangle cannot be negative, right?

In general, in doing such problems using algebra, it is a good practice to find both answers and then think back to the context of the original problem, to decide whether to take both solutions or only one.

One more problem:

- Of three consecutive integers, 1 added to the product of the first and the third gives 169. What are the numbers?

Can we do this using inverse operations? So, let's try algebra. Taking the numbers as x, $x + 1$ and $x + 2$, the given information translates to the equation

$$x(x + 2) + 1 = 169$$

This we can write as

$$x^2 + 2x + 1 = 169$$

What next?

Solutions—math and reality

We want to make a rectangle of perimeter 20 centimetres and height 11 centimetres less than the width.

Taking the width as x, the algebraic form of this problem is

$$x + (x - 11) = 10$$

which gives

$$x = 10.5$$

That is, the width should be 10.5 centimetres. What about the height then?

$$10 - 10.5 = -0.5$$

which is impossible.

Now look at this problem:

The distance between two numbers marked on the number line is 11; and their sum is 10. What are the numbers?

Taking the larger number as x, we get the same equation as in the first problem. The numbers turn out to be $10\frac{1}{2}$ and $-\frac{1}{2}$. And this is indeed a solution to this problem.

In general, the equations arising from different contexts may be the same; but the solutions may not suit all the contexts.

On squares

We have seen an algebraic identity about the square of a sum of two numbers, in Class 8.

For any two numbers x and y

$$(x + y)^2 = x^2 + 2xy + y^2$$

If we choose various numbers in the place of y in this, we get various identities, for all numbers x, such as

$$(x + 1)^2 = x^2 + 2x + 1$$

$$(x + 3)^2 = x^2 + 6x + 9$$

$$\left(x + \tfrac{2}{3}\right)^2 = x^2 + \tfrac{4}{3}x + \tfrac{4}{9}$$

$$(x - 4)^2 = x^2 - 8x + 16$$

Look again at the expressions on the right side of these equations. All are second degree polynomials. Notice any relation between the coefficient of x and the last number?

Can you write $x^2 + 8x + 64$ as the square of a first degree polynomial? What about $x^2 + 8x + 16$?

As in the previous problems, can we write the left hand side expression as a square?

Recall that

$$(x + 1)^2 = x^2 + 2x + 1$$

So, what can we do with our equation? We can write it as

$$(x + 1)^2 = 169$$

From this we can find

$$x + 1 = \pm\sqrt{169} = \pm 13$$

and so

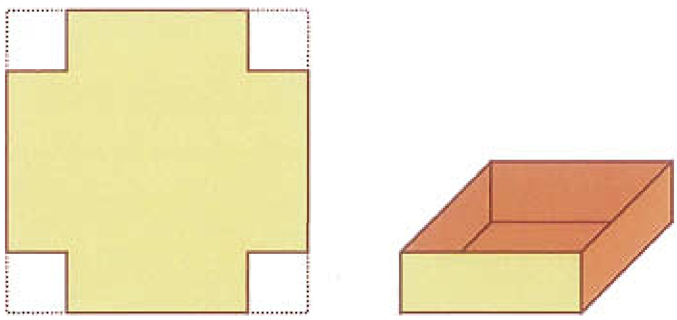
$$x = -1 \pm 13$$

$$x = 12 \text{ or } x = -14$$

Thus the numbers are 12, 13, 14 or −14, −13, −12.

Now try these problems on your own:

- 2000 rupees is invested in a scheme giving interest compounded annually. After two years the amount became 2205 rupees. What is the rate of interest?
- A pavement 2 metres wide runs around a square ground. The total area of the ground and the pavement is 1225 square metres. What is the area of the ground?
- A box is to be made by cutting off small squares from the corners of a square of thick paper and turning up the tabs.



The height of the box should be 5 centimetres and it should have a capacity of $\frac{1}{2}$ litre. What should be the length of a side of the square cut off? And the length of a side of the original square?

- The common difference of an arithmetic sequence is 1 and the product of the first and the third terms is 143. What are the first three terms?

Completing the square

How did you do the last problem above?

If we take the second term of the sequence as x, then the first and the third terms would be $x - 1$ and $x + 1$; and from the given facts, we get

$$(x - 1)(x + 1) = 143$$

In this,

$$(x - 1)(x + 1) = x^2 - 1$$

So, the equation becomes

$$x^2 - 1 = 143$$

From this we get

$$x^2 = 144$$

$$x = \pm 12$$

and the first three terms of the sequence as 11, 12, 13 or -13, -12, -11.

Instead of this method, how about taking the first term of the sequence as x? Then we get the equation

$$x(x + 2) = 143$$

and we can write this as

$$x^2 + 2x = 143$$

What do we do next?

Can we write the polynomial on the left side of this equation as a square? Take a look at the earlier problems again. We know that

$$x^2 + 2x + 1 = (x + 1)^2$$

So, what if we add 1 to both sides of our equation? We get

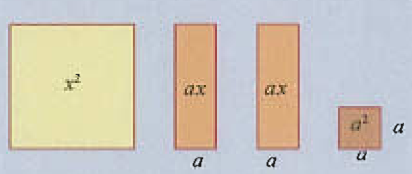
$$x^2 + 2x + 1 = 144$$

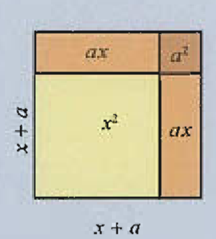
Geometry and algebra

For any two numbers x and a, we have

$$x^2 + 2ax + a^2 = (x + a)^2$$

For positive numbers x and a, this can be geometrically represented as below:

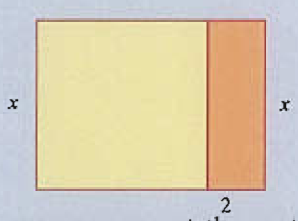




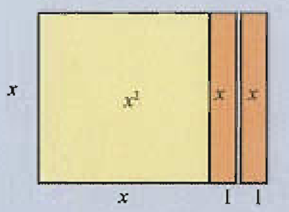
Geometric square completion

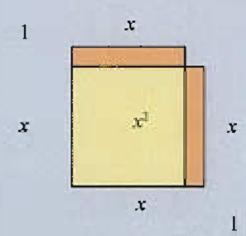
The algebraic fact that we can make $x^2 + 2x$ the square of an expression by adding 1 can be visualized geometrically also.

For this, we must first see $x^2 + 2x$ as a square of side x and a rectangle of side x and 2, joined together.

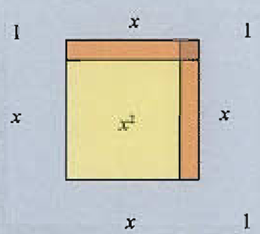


Now suppose we cut the rectangle into two equal pieces and shift one piece upwards as below:





To make this figure a square of side $x + 1$, all we need is a square of side 1 at the top right corner, right?



That is,

$$(x + 1)^2 = 144$$

The rest is easy, isn't it?

$$x + 1 = \pm\sqrt{144} = \pm 12$$

$$x = -1 \pm 12$$

$$x = 11 \text{ or } x = -13$$

And we get the terms of the sequence as 11, 12, 13 or −13, −12, −11.

Let's slightly modify this problem:

- The common difference of an arithmetic sequence is 6 and the product of the first and the second terms is 280. What are the first three terms of this sequence?

Taking the first term as x, we get the equation

$$x^2 + 6x = 280$$

In this, to write the left hand side of the equation in the form of a square, what number should we add? (Recall that for any two numbers x and a, we have $(x + a)^2 = x^2 + 2ax + a^2$).

We can write $x^2 + 6x$ as $x^2 + (2 \times 3)x$. So, to change $x^2 + 6x$ into $(x + 3)^2$, we need only add $3^2 = 9$. Thus we change our equation as

$$x^2 + 6x + 9 = 289$$

which means

$$(x + 3)^2 = 289$$

Now we can find

$$x + 3 = \pm 17$$

$$x = 14 \text{ or } -20$$

So, the first three terms of the sequence are 14, 20, 26 or −20, −14, −8

Let's look at another problem:

- A mathematician travelled 300 kilometres to attend a conference. During his talk he said, "Had the average speed of my trip been increased by ten kilometres per hour, I would have reached here one hour earlier". What was the average speed of his trip?

Taking the average speed as x kilometres per hour, what is the time taken for the trip? $\frac{300}{x}$.

If the speed is increased by 10 kilometres per hour, what would be the time taken for the trip? $\frac{300}{x+10}$.

So, what the mathematician said can be written as the equation.

$$\frac{300}{x} - \frac{300}{x+10} = 1$$

We can change this as below:

$$\frac{300(x+10) - 300x}{x(x+10)} = 1$$

$$\frac{300(x+10-x)}{x(x+10)} = 1$$

From this, we get

$$x(x + 10) = 3000$$

That is,

$$x^2 + 10x = 3000$$

Now to change the polynomial on the left to a square, what must we add?

So, we write this equation as

$$x^2 + 10x + 25 = 3025$$

That is,

$$(x + 5)^2 = 3025$$

From this, can't we calculate the average speed as 50 kilometres per hour?

Different way

There is another method to solve the equation, $x(x + 6) = 280$. We write $x + 6$ as $(x + 3) + 3$ and x as $(x + 3) - 3$. Using these, we find

$$\begin{aligned} x(x+6) &= \big((x+3)-3\big)\big((x+3)+3\big) \\ &= (x+3)^2 - 3^2 \end{aligned}$$

So, the original equation can be written

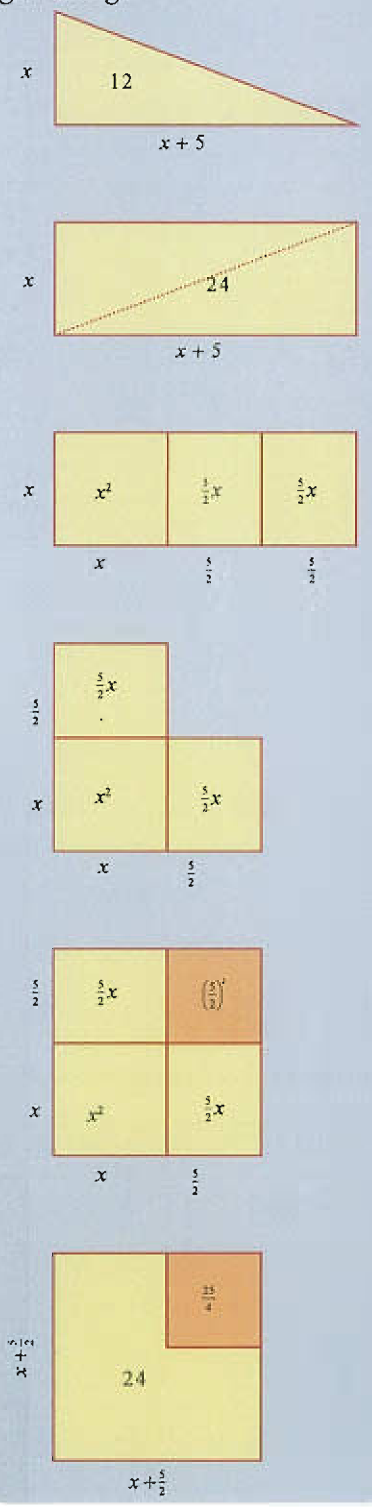
$$(x + 3)^2 - 9 = 280$$

From this we can find x as we did earlier.

See if you can solve $x^2 + 10x = 3000$ this way.

Geometry of solution

See the geometric explanation of the solution to the problem on a right angled triangle:



Now another problem:

- In a right angled triangle, one of the perpendicular sides is 5 centimetres longer than the other and its area is 12 square centimetres. What are the lengths of its sides?

If we take the shorter of the perpendicular sides to be x centimetres long, then the other is $x + 5$ centimetres long. What about the area?

Thus the given facts, translated to algebra becomes

$$\frac{1}{2}x(x+5) = 12$$

and this we can rewrite as

$$x^2 + 5x = 24$$

What number can we add to $x^2 + 5x$ to get an expression of the form $x^2 + 2ax + a^2$?

Let's first write

$$x^2 + 5x = x^2 + \left(2 \times \frac{5}{2}\right)x$$

And note that

$$x^2 + 2 \times \left(\frac{5}{2}\right) x + \left(\frac{5}{2}\right)^2 = \left(x + \frac{5}{2}\right)^2$$

So, let's add $\left(\frac{5}{2}\right)^2$ to either side of our equation of the problem.

This gives

$$x^2 + 2 \times \left(\frac{5}{2}\right) x + \left(\frac{5}{2}\right)^2 = 24 + \frac{25}{4}$$

That is,

$$\left(x + \frac{5}{2}\right)^2 = \frac{121}{4}$$

From this we find

$$x = -\frac{5}{2} \pm \frac{11}{2}$$

and this gives

$$x = -\frac{5}{2} + \frac{11}{2} \text{ or } x = -\frac{5}{2} - \frac{11}{2}$$

Here, $-\frac{5}{2} - \frac{11}{2}$ is a negative number. Since a length cannot be negative, this doesn't work for our problem. So, we need to take only $x = -\frac{5}{2} + \frac{11}{2} = 3$

Thus the lengths of the perpendicular sides of our triangle are 3 and 8 centimetres. From these, can't you find the hypotenuse? Do it.

Just one more problem:

- A rectangle is to be made with perimeter 100 centimetres and area, 525 square centimetres. What should be the lengths of its sides?

The sum of the lengths of any two adjacent sides of this rectangle is 50 centimetres, isn't it? (How?) So, if we take the length of one side as x, then the length of the other is $50 - x$. The area is to be 525 square centimetres. So,

$$x(50 - x) = 525$$

and this can be written as

$$50x - x^2 = 525$$

It's more convenient to change this slightly and write

$$x^2 - 50x = -525$$

(Why?) Now what? If we recall

$$x^2 - 2ax + a^2 = (x - a)^2$$

Things become easy. What number do we add to $x^2 - 50x$?

Thus we can change our equation to

$$x^2 - 50x + 25^2 = -525 + 25^2$$

That is,

$$(x - 25)^2 = 100$$

From this, we get

$$x = 25 \pm 10$$

Increase and decrease

There's another way to find a rectangle with perimeter 100 centimetres and area 525 square centimetres.

A square of perimeter 100 centimetres has sides 25 centimetres long and thus area 625 square centimetres. The area of the rectangle we seek is (naturally) less than this. To get this rectangle, we shorten one side of this square; to keep the perimeter the same, we have to lengthen the other side by the same amount.

If the decrease (and increase) is taken as x centimetres, then the lengths of the sides of our rectangle are $25 - x$ and $25 + x$. So, the equation of our problem is

$$(25 - x)(25 + x) = 525$$

That is

$$625 - x^2 = 525$$

From this we get

$$x = \pm 10$$

So, the lengths of the sides of the rectangle are $25 - 10 = 15$ and $25 + 10 = 35$ centimetres.

That is,

$$x = 35 \text{ or } x = 15$$

If we take $x = 35$, then $50 - x = 15$ and if $x = 15$, then $50 - x = 35$. Either way, the lengths of the sides of the rectangle are 35 and 15 centimetres.

Take a look at the algebraic forms of all these problems:

- $x^2 + 2x = 143$
- $x^2 + 10x = 3000$
- $x^2 + 5x = 24$
- $x^2 - 50x = -525$

The general form of all these is

$$x^2 + ax = b$$

Next see how we changed these equations:

- $x^2 + 2x + 1^2 = 143 + 1^2$, that is, $(x + 1)^2 = 144$
- $x^2 + 10x + 5^2 = 3000 + 5^2$, that is, $(x + 5)^2 = 3025$
- $x^2 + 5x + \left(\frac{5}{2}\right)^2 = 24 + \left(\frac{5}{2}\right)^2$, that is, $\left(x + \frac{5}{2}\right)^2 = \frac{121}{4}$
- $x^2 - 50x + 25^2 = -525 + 25^2$, that is, $(x - 25)^2 = 100$

The general method is to make $x^2 + ax$ a square, by adding the square of half the coefficient of x:

$$x^2 + ax + \left(\frac{a}{2}\right)^2 = \left(x + \frac{a}{2}\right)^2$$

This method is called *completing the square*.

Look at this problem:

- How many terms of the arithmetic sequence 3, 7, 11, . . . must be added to get 300?

Denoting the terms of the sequence as $x_1, x_2, x_3, \ldots$, we have

$$x_n = 3 + 4(n - 1) = 4n - 1$$

So,

$$x_1 + x_2 + x_3 + \ldots + x_n = \frac{1}{2}n(x_1 + x_n)$$

Past and present

The method of completing squares to solve second degree equations is pretty old. We see the Babylonians using this technique to solve problems related to areas, as early as 1500 BC.

But the current practice of converting problems to algebraic equations was not used then. (This is at most five hundred years old.) Problems were formulated, and solutions described in ordinary language. In geometric problems, the methods used were also geometrical.

Thus from the historical point of view, what we now describe as geometric interpretations of algebraic methods were actually the motivation for the algebraic techniques developed later.

$$= \frac{1}{2}n(3 + (4n - 1))$$

$$= 2n^2 + n$$

What if we want this sum to be 300? We must have

$$2n^2 + n = 300$$

Is this of the form $x^2 + ax = b$, we have discussed?

How do we change the coefficient of n^2 to 1?

How about dividing both sides of our equation by 2?

$$n^2 + \frac{1}{2}n = 150$$

Next we complete the square:

$$n^2 + \frac{1}{2}n + \left(\frac{1}{4}\right)^2 = 150 + \frac{1}{16}$$

That is,

$$\left(n+\frac{1}{4}\right)^2 = \frac{2401}{16}$$

Now can't we find n?

$$n = -\frac{1}{4} \pm \frac{49}{4}$$

$$n = 12 \text{ or } n = -\frac{50}{4}$$

Since n is a natural number in our problem, we need only the solution $n = 12$.

So, adding 12 terms of this sequence would give 300.

Thus in some equations, we have the additional job of changing the coefficient of x^2 to 1, before completing the square.

Now try these problems:

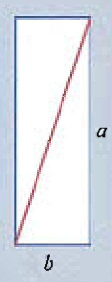
- The length of a rectangle is 10 centimetres more than its breadth; and its area is 144 square centimetres. What are the length and breadth?
- How many terms of the arithmetic sequence 5, 7, 9, . . . should be added to get 140?

Diagonal problem—algebra

We see that the method of completing squares was used not only for solving second degree equations, but for computing square roots also, in olden times.

For example, in an ancient Babylonian clay tablet, the method of computing the diagonal of a tall, slender rectangle is given like this:

divide the square of the width by the height and add half of this to the height



In the current algebraic language, this becomes

$$\sqrt{a^2 + b^2} \approx a + \frac{b^2}{2a}$$

Its rationale can also be explained with algebra, we know that

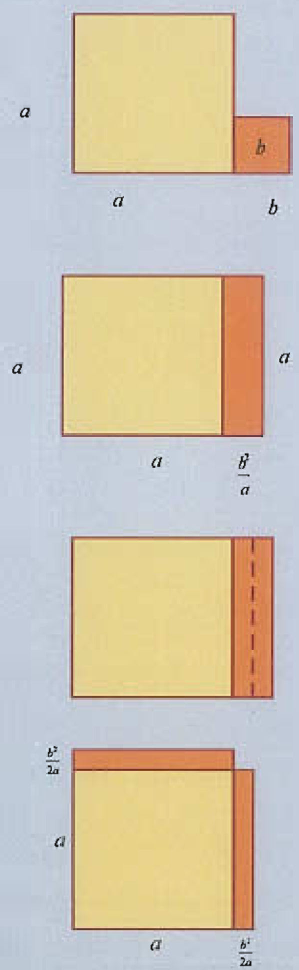
$$a^2 + b^2 + \left(\frac{b^2}{2a}\right)^2 = \left(a + \frac{b^2}{2a}\right)^2$$

If the number b is small compared to the number a, then $\left(\frac{b^2}{2a}\right)^2$ would be a very small number and so we can take

$$a^2 + b^2 \approx \left(a + \frac{b^2}{2a}\right)^2$$

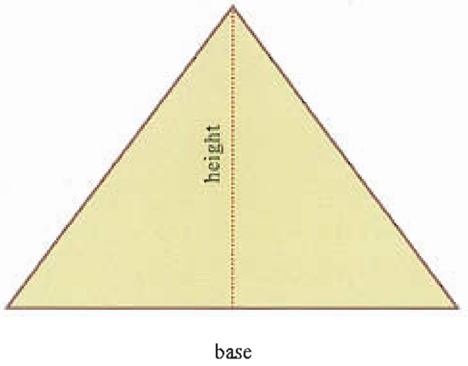
Diagonal problem—geometry

Let's have a look at the geometry of the Babylonian diagonal estimate:



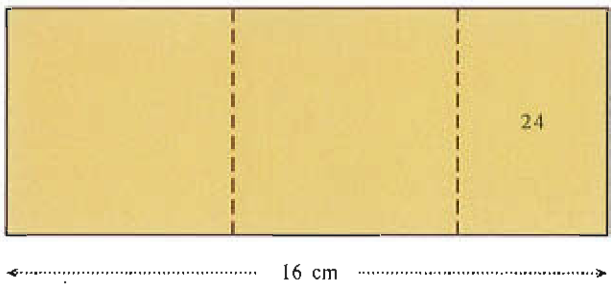
The last shape differs only slightly from a square of side $a + \frac{b^2}{2a}$, right?

- An isosceles triangle as shown below is to be made:



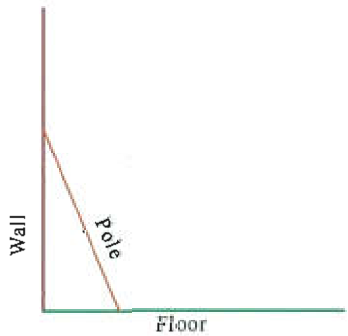
The height should be 2 metres less than the base and the area should be 12 square metres. What should be the lengths of the sides?

- From a rectangular sheet of paper, two squares are cut off as shown below:



The remaining rectangle should have area 24 square centimetres. What should be the lengths of the sides of the squares?

- A pole 2.6 metres high leans against a wall, with its foot 1 metre away from the wall.



When this distance was slightly increased, the top of the pole slid down the wall by the same distance. How far was the foot of the pole shifted?

- The perimeter of a rectangle is 28 metres and its diagonal is 10 metres. What are the lengths of its sides?

- Find a pair of numbers with sum 4 and product 2.

- The sum of a number and its reciprocal is $2\frac{1}{12}$. What are the numbers?

- Thirty candies were distributed among some children. Relishing the sweet, the whiz-kid among them said, "If we were one short, we would all get one more." How many kids were there?

- There are two taps opening into a tank. If both are opened, the tank would be full in 12 minutes. The time taken for it to fill with only the small tap open, is 10 minutes more than the time to fill with only the large tap open. What is the time taken to fill the tank with only the small tap open?

Equations and polynomials

The general algebraic form of all the problems done so far is

$$ax^2 + bx + c = 0$$

For example, our very first problem gave the equation

$$(x + 5)^2 = 36$$

and this we can write

$$x^2 + 10x - 11 = 0$$

Another equation we came across in one of our problems is

$$2x^2 + x = 300$$

and we can write this as

$$2x^2 + x - 300 = 0$$

Looking at it like this, another idea takes shape. We know that $ax^2 + bx + c$, with $a \neq 0$ is the general form of a second degree polynomial.

So all the problems we have done can be seen as investigations in

Square root

We can use the Babylonian technique (finding the diagonal of a rectangle), for computing approximations of square roots also. For any two numbers x and a, we have

$$a^2 + x + \left(\frac{x}{2a}\right)^2 = \left(a + \frac{x}{2a}\right)^2$$

If the number x is small compared to a, we can take

$$a^2 + x \approx \left(a + \frac{x}{2a}\right)^2$$

and so

$$\sqrt{a^2 + x} \approx a + \frac{x}{2a}$$

For example, since $2 = \frac{9}{4} - \frac{1}{4}$, we have

$$\begin{aligned}\sqrt{2} &= \sqrt{\left(\frac{3}{2}\right)^2 - \frac{1}{4}} \\ &\approx \frac{3}{2} - \frac{1}{2} \times \frac{1}{4} \times \frac{2}{3} \\ &= \frac{17}{12} \\ &\approx 1.4166\end{aligned}$$

We have seen in Class 9 that $\frac{17}{12}$ was used as an approximation for $\sqrt{2}$ in ancient Babylon and in our place also (the sections **An old method** and **Folk math** of the lesson **Irrational Numbers**).

Polynomials

We can use the method of completing squares to find out certain features of a second degree polynomial.

For example, consider the polynomial

$$p(x) = x^2 + 6x + 11$$

By the process of completing squares, we can write

$$\begin{aligned} p(x) &= x^2 + 6x + 11 \\ &= (x^2 + 6x + 9) + 2 \\ &= (x + 3)^2 + 2 \end{aligned}$$

In this, whatever number we take as x, the number $(x + 3)^2$ would not be negative. So, the number got as $p(x)$ would never be less than 2.

In other words, if we take various numbers in the place of x and calculate the numbers got as $p(x)$, the least number got would be 2; and this is got when we take $x = -3$.

finding the numbers for which a second degree polynomial gives zero.

Now let's see the algebraic form of our method of solution also. To solve the equation

$$ax^2 + bx + c = 0$$

we first rewrite the equation in the more familiar form

$$ax^2 + bx = -c$$

Then we divide both sides by a to get

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Next to change the left side expression to a square, we add the square of half the coefficient $\frac{b}{a}$ of x:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

The left side of this can be written

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2$$

and the right side can be simplified to

$$-\frac{c}{a} + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

Thus our equation becomes

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

From this we get

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

and then

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus we see that

For any three numbers a, b, c with $a \neq 0$, the solution to the equation

$$ax^2 + bx + c = 0 \quad is \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In view of our earlier discussion, this can also be stated as follows.

In the polynomial

$$p(x) = ax^2 + bx + c$$

if we take

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

then

$$p(x) = 0$$

In solving second degree equations, instead of dividing by the coefficient of x and then completing the square, we can directly use this formula.

For example, consider the equation

$$2x^2 + x = 300$$

Writing this as

$$2x^2 + x - 300 = 0$$

We can find x directly as

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \times 2 \times (-300)}}{2 \times 2}$$

$$= \frac{-1 \pm \sqrt{2401}}{4}$$

$$= \frac{-1 \pm 49}{4}$$

$$= 12 \text{ or } \frac{25}{2}$$

Here are some problems for you.

Find the answers to the following problems correct to two decimal places using a calculator.

Different methods

To solve a second degree equation, we can do the process of completing the square step by step; or we can use the formula which does all this work in one stroke and quickly gives the solution. The convenience in using either depends on the nature of the equation.

For example, to solve

$$x^2 + 12x + 7 = 0$$

completing the square to get

$$(x + 6)^2 = -7 + 36$$

and proceeding may be better than directly trying to compute

$$x = \frac{-12 \pm \sqrt{12^2 - 4 \times 1 \times 7}}{2}$$

On the other hand, for an equation like

$$2x^2 + 5x - 3 = 0$$

directly computing

$$x = \frac{-5 \pm \sqrt{5^2 + 4 \times 2 \times 3}}{4}$$

may be better than completing the square to get

$$2\left(x + \frac{5}{4}\right)^2 = -3 + \frac{25}{8}$$

- A rectangle of perimeter 10.75 metres and area 5.8 square metres is to be made. What should be the lengths of its sides?
- The distance travelled by an object thrown upwards in t seconds is $30t - 4.9t^2$ metres. After how much time would it fall down? At what all times would it be 20 metres above the ground?
- In each of the second degree polynomials given below, find the numbers to be taken as x to get zero:
  - $x^2 - 5x + 6$
  - $x^2 - 2x - 1$
  - $x^2 + 5x + 6$
  - $2x^2 - 7x - 15$
  - $x^2 + x - 6$
  - $9x^2 + 12x + 4$
  - $x^2 - x - 6$

Discrimination

Armed with our new knowledge, let's tackle an old problem:

- We want to make a rectangle of perimeter 20 centimetres and area 26 square centimetres. What should be the lengths of the sides?

The equation of these requirements is

$$x(10 - x) = 26$$

and we can write this as

$$x^2 - 10x + 26 = 0$$

Trying out our new technique, we get

$$x = \frac{10 \pm \sqrt{100 - 104}}{2} = \frac{10 \pm \sqrt{-4}}{2}$$

What does $\sqrt{-4}$ mean? Negative numbers don't have square roots!

The meaning of this answer is that there are no numbers x for which the equation $x^2 - 10x + 26 = 0$ holds. (In other words, whatever number x we take, we won't get zero from the the polynomial $x^2 - 10x + 26$.)

In general, note that in the solution for the equation, $ax^2 + bx + c = 0$, we have a square root, $\sqrt{b^2 - 4ac}$. If $b^2 - 4ac$ is a positive number, then it has two square roots and each leads to a solution of the equation.

Squares again!

We have seen in Class 9 that among all rectangles of perimeter 20 centimetres, the square of side 5 centimetres has the maximum area (the section **Square speciality.** of the lesson **Polynomials**).

This we can see in a different way. If the length of one side of such a rectangle is taken as x, then its area can be found from the polynomial

$$p(x) = x(10 - x) = 10x - x^2 = -(x^2 - 10x)$$

Completing the square, we can write

$$p(x) = -\left((x - 5)^2 - 25\right) = 25 - (x - 5)^2$$

In this, whatever be the number we take as x, the number $(x - 5)^2$ would not be negative; so that the number $p(x)$ would not be greater than 25. And for $x = 5$, we do get $p(x) = 25$

On the other hand, if $b^2 - 4ac$ is negative, then the equation has no solution.

What if $b^2 - 4ac$ is zero? Then it has only one square root (zero itself) and so the equation has only one solution.

The number $b^2 - 4ac$ is called the *discriminant* of the equation $ax^2 + bx + c = 0$. So, what we found out is this:

> *If the discriminant of a second degree equation is positive, then it has two solutions; if it is negative, the equation has no solutions; and if it is zero, the equation has only one solution*

Now look at this problem:

- An 8 centimetres long wire is to be bent into a rectangle. Can a rectangle with diagonal 2 centimetres be made from it? What about a rectangle with diagonal 4 centimetres?

Taking the length of one side of such a rectangle as x the other must be $4 - x$ and so the square of the diagonal must be

$$x^2 + (4 - x)^2 = 2x^2 - 8x + 16$$

So, the first question is whether this can be 4. That is,

$$2x^2 - 8x + 16 = 4$$

This we can write as

$$2x^2 - 8x + 12 = 0$$

The discriminant of this equation is

$$(-8)^2 - 4 \times 2 \times 12 = 64 - 96 < 0$$

So, it is not possible to make such a rectangle.

Next, let's check whether a rectangle of diagonal 4 centimetres is possible. The equation of this wish is

$$2x^2 - 8x = 0$$

and its discriminant is

$$(-8)^2 - 4 \times 2 \times 0 = 8^2 = 64$$

Word and meaning

The word *discriminant* in ordinay language means a feature which makes something different from others. And *discrimination* means understanding of the difference between one thing and another, and also good judgement and taste.

It is also used to mean unjust treatment of different groups of people.

Polynomial and discriminant

We can use the discriminant to understand some features of a second degree polynomial also. In the polynomial

$$p(x) = ax^2 + bx + c$$

we can complete the square and write

$$p(x) = a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right)$$

$$= a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right)$$

In this, $\left(x + \frac{b}{2a}\right)^2$ is not negative, whatever be the number x. Also, if $b^2 - 4ac$ is negative, then $\frac{4ac - b^2}{4a^2}$ is positive. So, if a is positive, then $p(x)$ is positive for any number x and if a is negative, then so is $p(x)$, for any number x.

What do we see from this? If the discriminant $b^2 - 4ac$ is negative, then depending on whether a is positive or negative, the numbers got from the polynomial $p(x)$ would be all positive or all negative.

What if $b^2 - 4ac$ is positive? And what if it is zero?

So, the equation has two solutions and they are

$$x = \frac{8 \pm \sqrt{64}}{4} = 4 \text{ or } 0$$

Since x is the length of the side of a rectangle in our problem, $x \neq 0$. If we take $x = 4$, then the length of the other side would be $4 - x = 0$. So, either way, we cannot make such a rectangle.

What do we see here? Even if the mathematical equation arising from a physical problem has a solution, the physical problem itself may not have a solution.

Now try to answer the questions below:

- Can the sum of the first few consecutive terms of the arithmetic sequence 5, 7, 9, . . . be 140? What about 240?
- From the polynomial $p(x) = x^2 + x + 1$, do we get $p(x) = 0$ Can the sum be 240? any x? What about $p(x) = 1$? And $p(x) = -1$?
- From the expression $x + \frac{1}{x}$, do we get 0, 1, or 2 for some number x?
- Prove that if a, b, c are positive numbers and if the equation $ax^2 + bx + c = 0$ has solutions, then they are negative numbers.