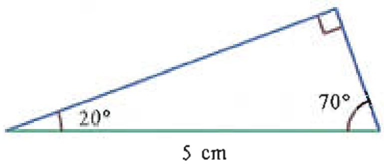
2 Circles

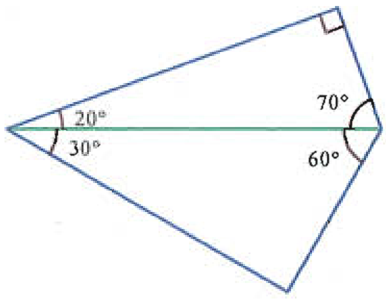
Angle drawings

We want to draw a right angled triangle of hypotenuse 5 centimetres; the other two sides of any length as we please. In what all ways can we do this?

We can start with a line 5 centimetres long. Draw any angle we please at one end. Subtract this angle from 90 and draw the angle thus got at the other end, to make a triangle. For example:



We can also draw something like this below the line:

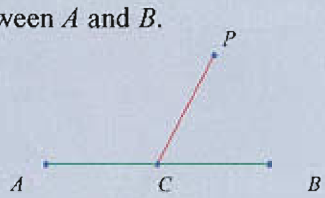


Another way is to use a set square from the geometry box. Place it with its right angle above (or below) the line and the perpendicular edges through the end-points of the line. Try!

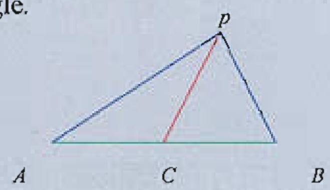
Draw many such triangles and look at their third vertices:

Right angles from circles

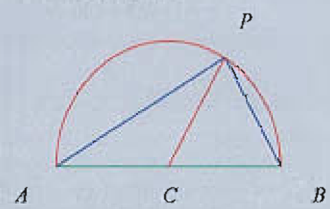
There is yet another way to draw a right-angled triangle with a specified hypotenuse AB. Mark the mid-point C of AB and mark a point P at a distance equal to half the distance between A and B.



We can show that APB is a right angle.

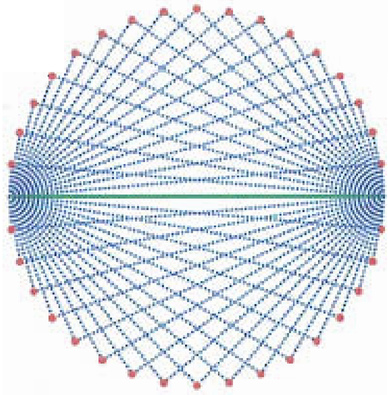


Since $CA = CB = CP$, the circle of radius this length, centred at C, passes through A, B and P

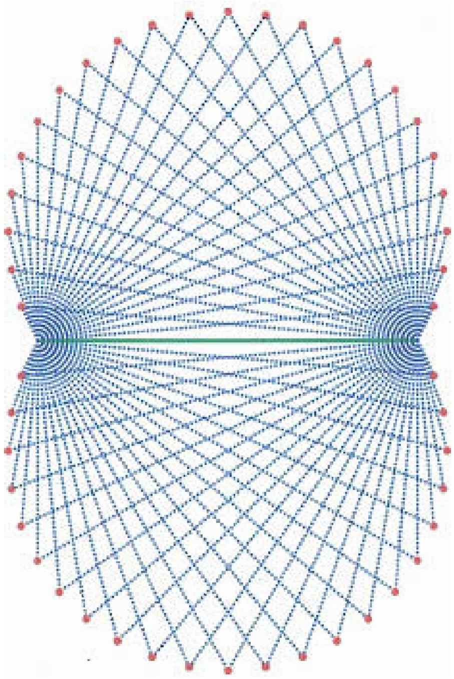


So, $\angle APB$ must be equal to 90°.

(Recall the section **Angle in a semicircle**, of the lesson **Congruent Triangles** in the Class 8 textbook.)

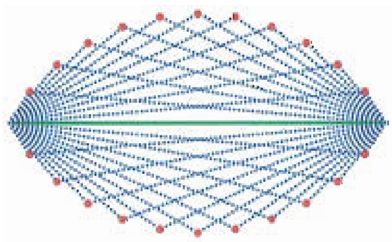


Now instead of a right angle, let's draw triangles with one angle 60° (We can use a corner of a set square.)



How about a 45° angle or a 30° angle? These can also be done with set squares.

Now cut out a triangle with a 120° angle from thick paper. Use it to draw pictures like the above with the top angles 120°:



Why do we get pictures like these? Let's investigate.

Circles from right angles

We saw that if we draw a circle with AB as diameter, then taking any point P on this circle, we could draw a right angled triangle APB.

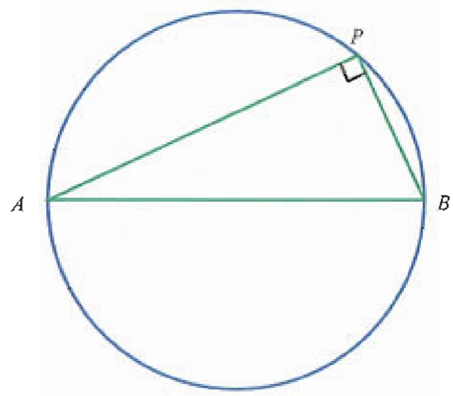
On the other hand, if we draw a right angled triangle with AB as hypotenuse, then the diameter of the circumcircle of ΔAPB would be AB. (There is such a problem in the section **Another division,** of the lesson **Geometric Proportions** in the Class 9 textbook)

So, if we draw all right angled triangles with AB as hypotenuse and take only their third vertices, we get all points on the circle with AB as diameter, except A and B.

Right angles and circles

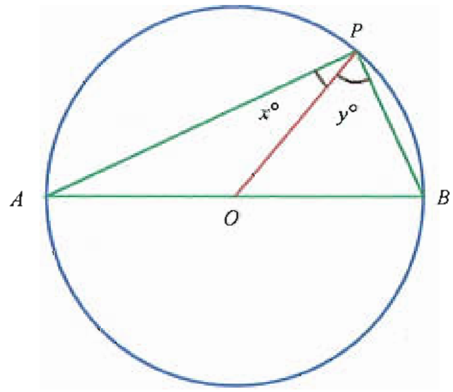
In the picture drawn with a right angle, we got a single circle and the line we started with turned out to be its diameter; that is, semicircles above and below this line.

Haven't we seen a picture like this before?

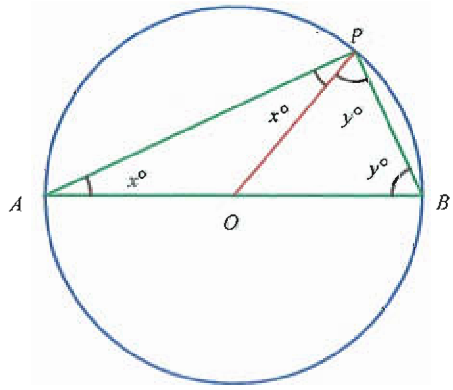


(Recall the section **Angle in a semicircle,** of the lesson **Congruent Triangles** in the Class 9 textbook.)

AB is a diameter of the circle. How did we prove that $\angle P$ is a right angle?



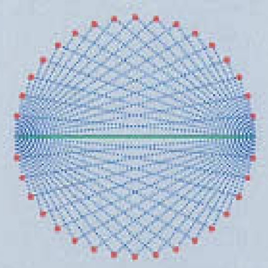
Here O is the centre of the circle. So, OAP and OBP are isosceles triangles. (Reason?) If we take $\angle APO = x°$ and $\angle BPO = y°$, then we get $\angle A = x°$ and $\angle B = y°$. (How is that?)



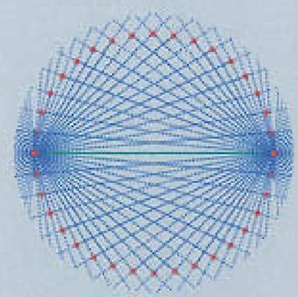
Locus

Locii of points can often be described in terms of lengths or angles. For example, we can think of the perpendicular bisector of the line joining two points as the locus of a point which moves, keeping the same distance from these points. We can also think of it as the locus of a point which moves such that the lines joining it with the end points make equal angles with the line.

What is the locus of the third vertex of a right angled triangle with a specified line as the hypotenuse? We saw that it is not the full circle with this line as diameter—the endpoints of the line are not on this locus.



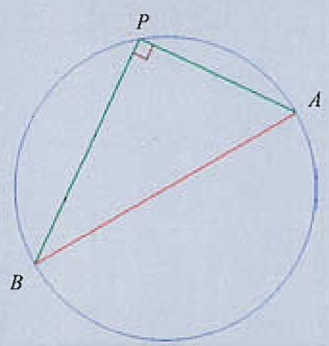
Instead, if we say the locus of the point of intersection of perpendicular lines through the end points of this line, then we get the entire circle.



Right angle and diameter

We saw that when two ends of a diameter of a circle are joined to another point on the circle, the angle formed at this point is a right angle.

On the other hand, suppose we draw a pair of perpendicular lines from a point on a circle and join the points of intersection of these lines with the circle. Is this line a diameter of the circle?



Here, the circle is the circumcircle of the right angled triangle APB. And we know that the hypotenuse of a right angled triangle is a diameter of its circumcircle. So, AB is indeed a diameter of the circle.

The section **Circle and square,** of the lesson **Between the lines** in the Class 7 textbook gives a trick to find the centre of a circle. Now do you see why this trick works?

Since the sum of the angles of ΔABP is 180°, we have

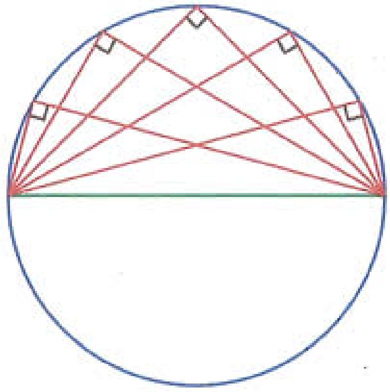
$$x + y + (x + y) = 180°$$

From this we get $2x + 2y = 180°$ and then

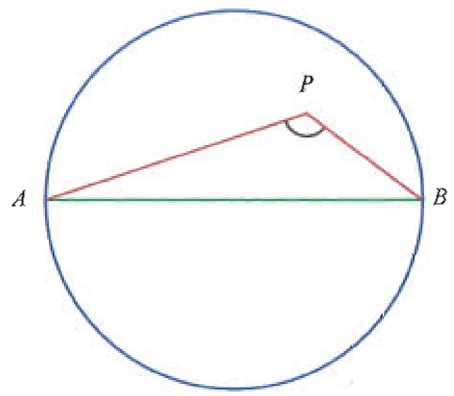
$$x + y = 90°$$

What do we see from this?

If the ends of a diameter of a circle are joined to any other point on the circle, then what we get is a right angle.

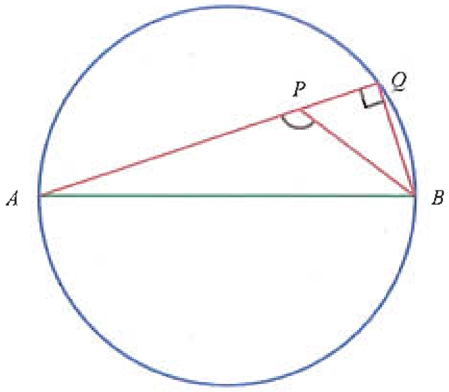


This leads to another thought. We get a right angle, when the ends of a diameter are joined to a point *on* the circle; what if we join them to a point *inside* the circle?



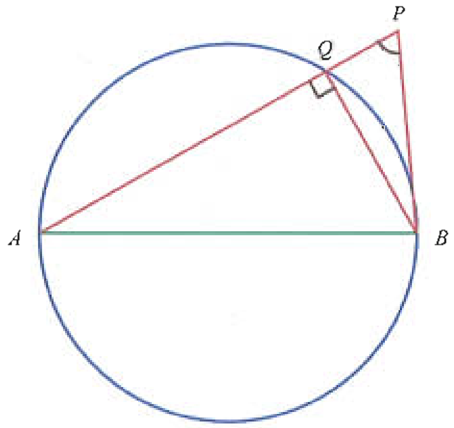
Do we get an angle larger than a right angle at every point within the circle?

In this picture, extend one line to meet the circle; join this point to the other end of the diameter.



Now $\angle APB$ is the exterior angle at the vertex P of ΔPQB. So, it is the sum of the (interior) angles of the triangle at Q and B. (This problem is in the section **Unchanging sums,** of the lesson **Polygons** in the Class 9 textbook.) Of these, the angle at Q is a right angle. This means $\angle APB$ is larger than a right angle, right?

Next, what about a point outside the circle?



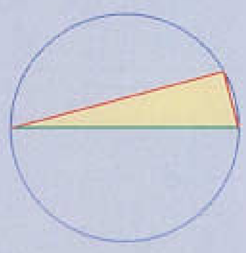
In this set up, $\angle APB$ is an internal angle of ΔPQB and $\angle AQB$ is an exterior angle. So, $\angle APB$ is smaller than a right angle.

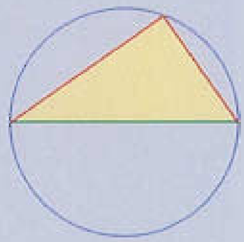
Now suppose that the ends of a diameter of a circle are joined to some point and we get a right angle at that point. This point cannot be *inside* the circle. (At points within the circle we get an angle larger than a right angle.) And it cannot be *outside* the circle. (For points outside the circle, we get an angle smaller than a right angle.)

So, the point must be *on* the circle. Don't you see now why we got a circle in our first picture drawn with a right angle? You can try some problems based on these ideas.

Square speciality

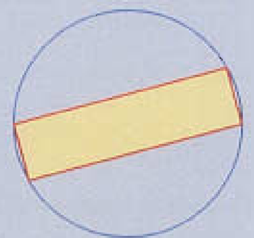
We can make a right angled triangle in a circle by joining various points of a circle to the ends of a diameter.

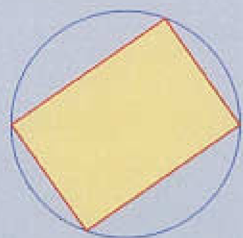




To get the maximum area, where should we choose the point?

This leads to another question. We can draw any number of rectangles with all four vertices on a circle:





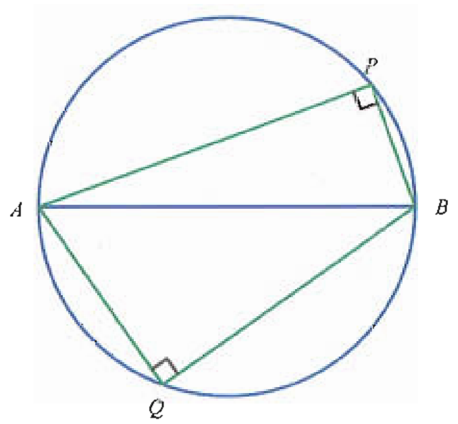
What is the speciality of the rectangle of maximum area?

- In ΔABC, we have $\angle A = 60°$ and $\angle B = 70°$. Is the vertex C inside or outside the circle with diameter AB?
- Prove that if a pair of opposite angles of a quadrilateral are right, then a circle can be drawn through all four of its vertices.
- In the quadrilateral $ABCD$, we have $AB = 3$cm, $BC = 4$ cm, $AC = 5$ cm, $\angle A = 120°$, $\angle C = 70°$. If we draw the circle with AC as diameter, which of the four vertices of $ABCD$ would be inside the circle? Which of them would be outside this circle? Is any vertex on the circle? What about the circle with BD as diameter?

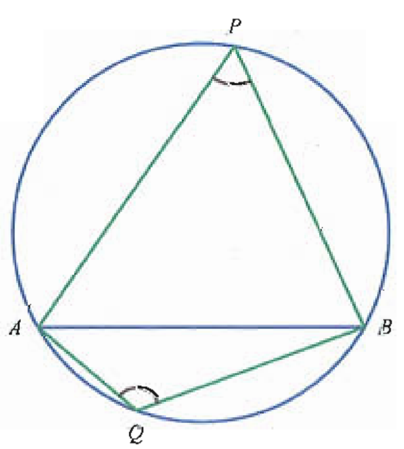
Angles, arcs and chords

We now know why we got a circle in the picture drawn with a right angle. What about the other pictures?

Again let's start with a circle. Any diameter of the circle divides it into two equal arcs; and we get a pair of right angles by joining points on each to the ends of the diameter.



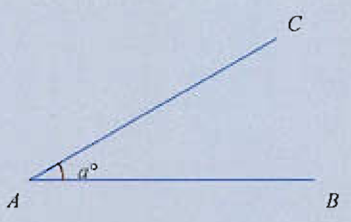
What about a chord which is not a diameter?



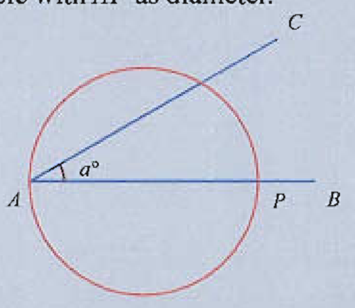
The arcs are not equal; and the angles are not right.

Doubling an angle

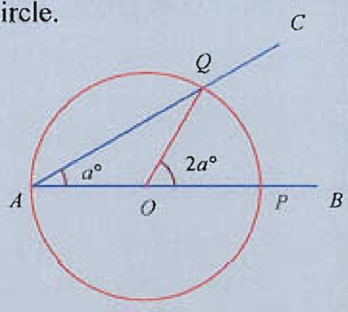
We know how we can halve an angle by drawing the bisector. How do we double an angle?



Mark a point P on AB and draw a circle with AP as diameter.

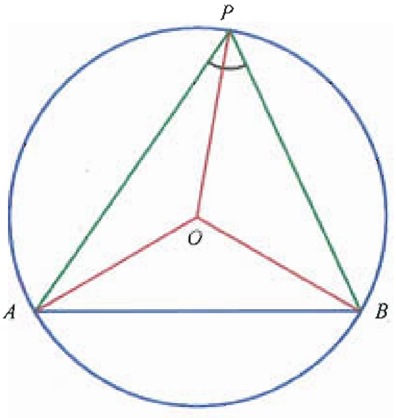


Join the point Q where this circle meets AC, and the centre O of the circle.

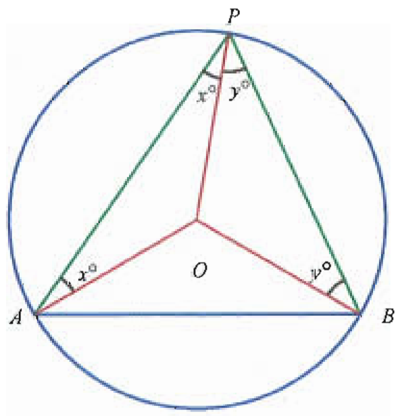


Since ΔOPQ is isosceles, $\angle OQA = a°$ and so the exterior angle POQ is equal to $2a°$.

Let's inspect the arc and angle at the top and bottom separately. First the upper ones. As we did in the case of a diameter, we join P to the centre O of the circle. Since the centre is not on the chord here, we also join OA and OB.

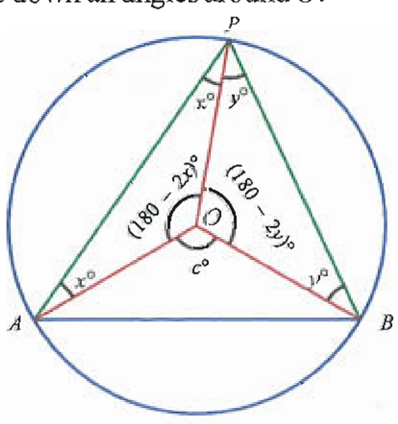


As in the case of diameter, here also, OAP and OBP are isosceles triangles.



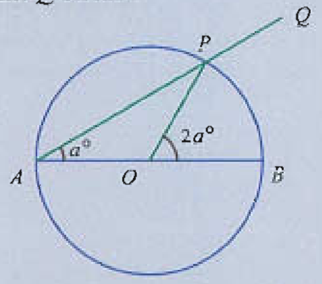
But here these two triangles do not form a single large triangle, as seen earlier. So, the old trick of summing up the angles of a triangle won't work.

Instead, let's write down all angles around O.



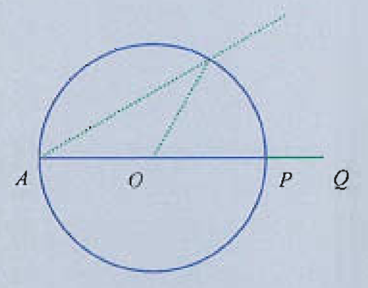
Angle of rotation

In the figure below, AB is a diameter of the circle and O is the centre. A point P on the circle is marked and a point Q on AP.

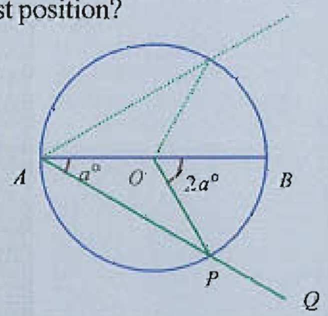


Let $\angle BAP = a°$, so that $\angle BOP = 2a°$.

Now suppose that P moves along the circle and reaches B.



The line OP has rotated through $2a°$. And the line AP through $a°$. What happens as P moves again and reaches the point directly below the first position?



Thus we get

$$(180 - 2x) + (180 - 2y) + c = 360$$

(Recall the section **Around a point,** of the lesson **Polygons** in the Class 9 textbook.) That is,

$$360 - 2(x + y) + c = 360$$

This gives

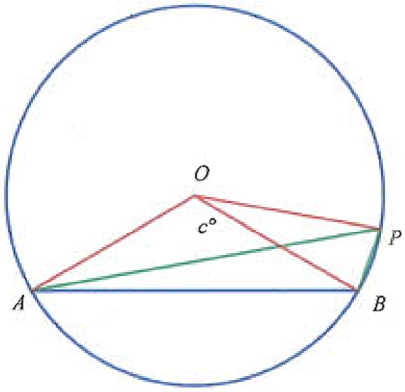
$$x + y = \frac{1}{2}c$$

That is,

$$\angle APB = \frac{1}{2}c°$$

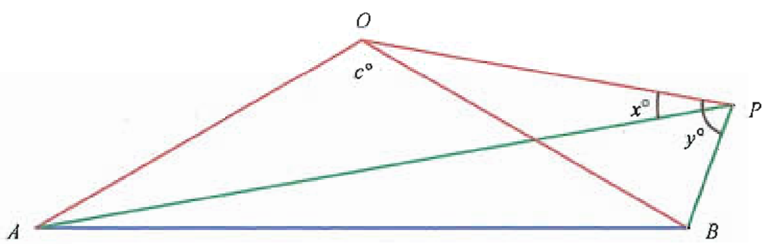
Would this be true, for any position of P on the upper arc?

What if it is like this?



Let's try writing $\angle OPA = x°$ and $\angle OPB = y°$ as before.

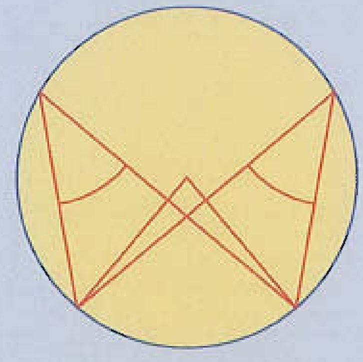
To see things clearly, let's enlarge the triangles in the picture:

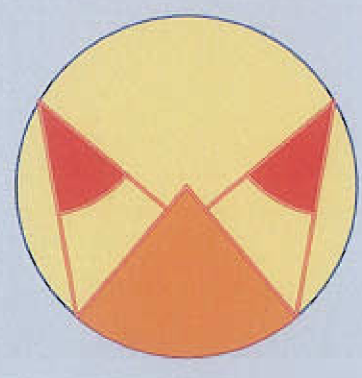


Now as before, we can find other angles, using the fact that ΔOAP and ΔOBP are isosceles.

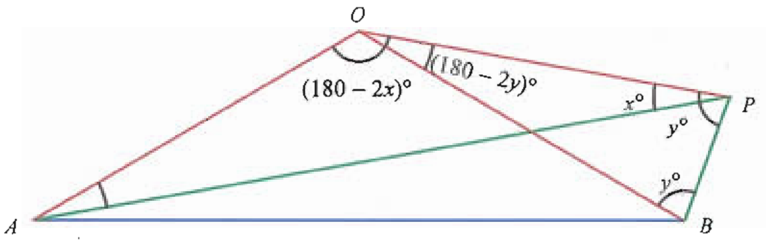
Cut and paste

Draw a figure and cut out pieces as shown below:





Now place them as shown below:



From the figure we see that

$$\angle APB = (y - x)^\circ$$

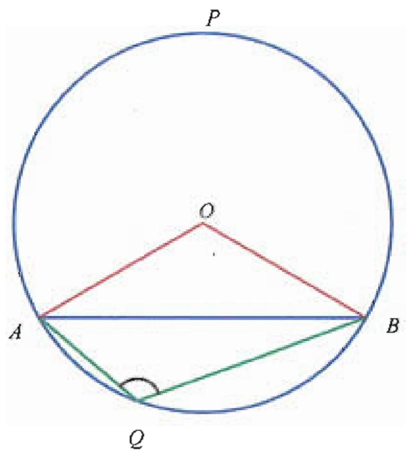
and that

$$\angle AOB = (180 - 2x) - (180 - 2y) = 2(y - x)^\circ$$

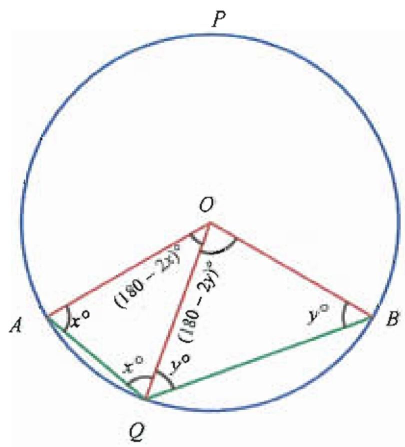
So, again we get

$$\angle APB = \frac{1}{2}c^\circ$$

What about angles below AB?

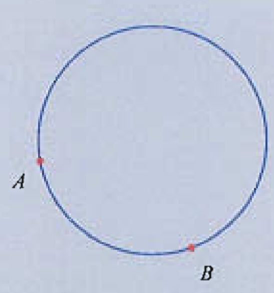


In this case also, by joining OQ, we get two isosceles triangles; and we can compute angles as we did earlier.



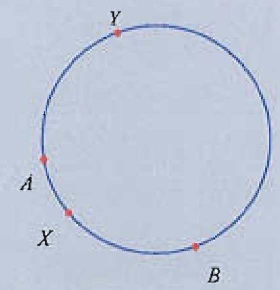
Arc pair

Two points on a circle actually divide it into two arcs:



In this picture, a small arc is got by travelling right from A to B along the circle and a large arc, by travelling left from A to B along the circle.

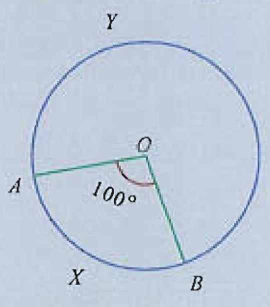
Choosing a point each on these arcs, we can name the arcs using the names of these points with the end points.



In this picture, the smaller arc is AXB and the larger arc is AYB.

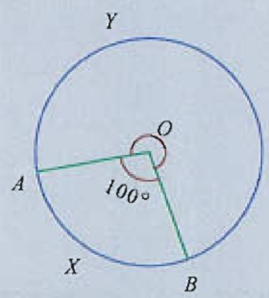
So, for any arc of a circle, there is a matching arc which completes it as a circle; and only one such arc. In other words, any arc can be completed as a circle in only one way.

Central angle



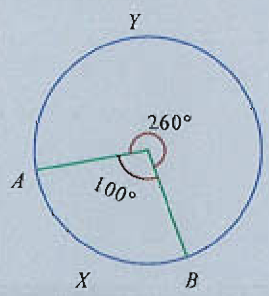
In the picture above, the central angle of the arc AXB is $100°$.

What about the central angle of the arc AYB?



By definition of the degree measure of an angle, the part $OAXB$ of the circle is made up of 100 out of 360 equal parts into which the circle is divided. So, how many parts make up the remaining part $OAYB$?

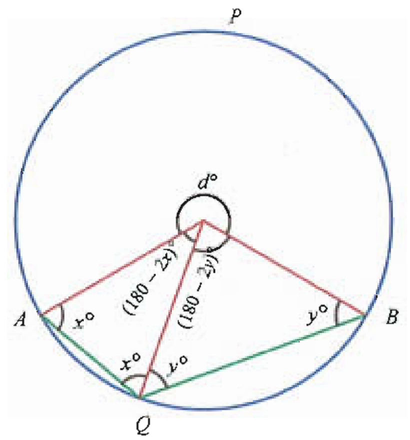
So, the central angle of the arc AYB is $260°$.



So, if we take the central angle of $\angle APB$ as d, then we can see that

$$(180 - 2x) + (180 - 2y) + d = 360$$

from the picture below:



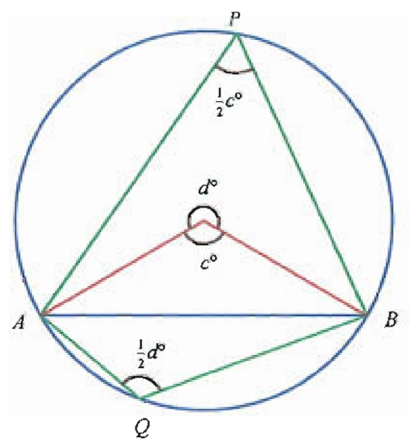
This equation gives

$$2(x + y) = d$$

That is,

$$\angle AQB = \frac{1}{2}d°$$

Let's sum up all we have seen so far. Look at this picture:

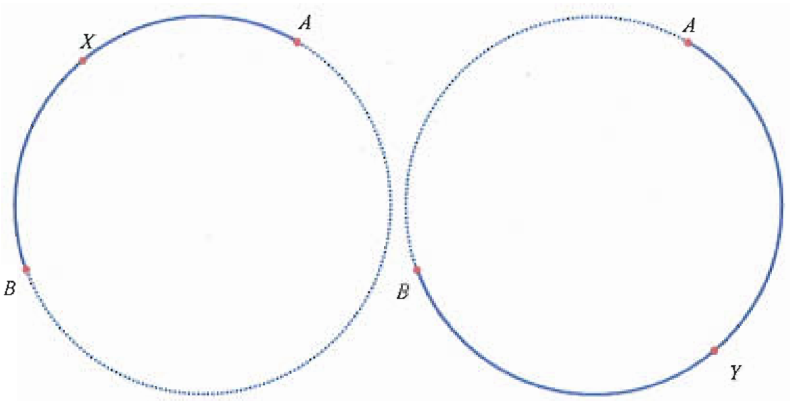


Wherever we take the point P, on the circle above AB, we get $\angle APB = \frac{1}{2}c°$.

Wherever we take the point Q, on the circle and below AB, we get $\angle AQB = \frac{1}{2}d°$.

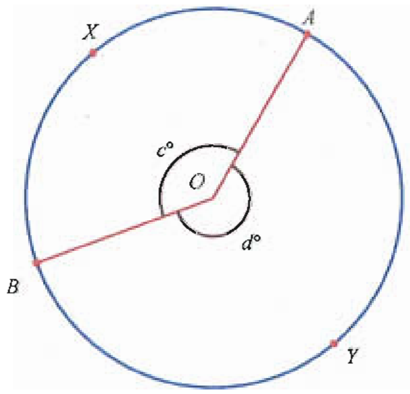
We can state this without referring to the chord AB.

Any two points on a circle divide it into a pair of arcs, right?

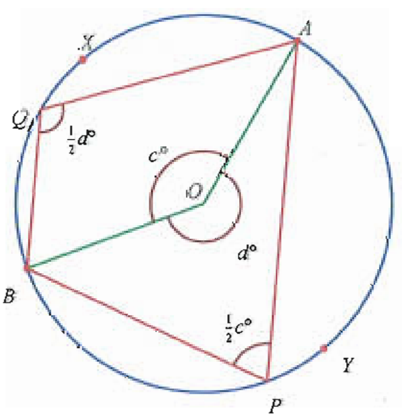


In this picture, the points A and B divide the circle into a pair of arcs AXB and AYB. The arc AYB is called the *alternate arc* (or the *complementary arc*) of AXB (and vice versa.)

Let's join A and B to the centre O of the circle.

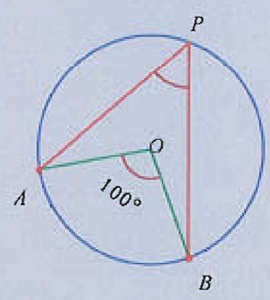


In this figure, $c°$ is the central angle of the arc AXB and $d°$ is the central angle of the arc AYB. Now let's take some point P on the arc AYB and some point Q on the arc AXB.



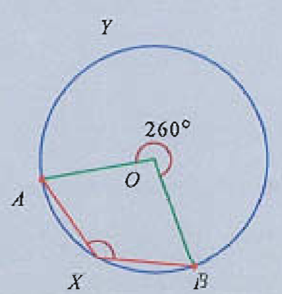
Then we can combine the two statements made earlier into one:

Changing angle



In the figure, $\angle APB = 50°$. Also, this angle would be the same $50°$, wherever we take P on the larger of the two arcs made by A and B.

Now suppose this point travels left along the circle. The angle doesn't change till it reaches A. At A, there is no angle to speak of. As it moves further left and moves into the smaller arc, the angle changes. What does it become?



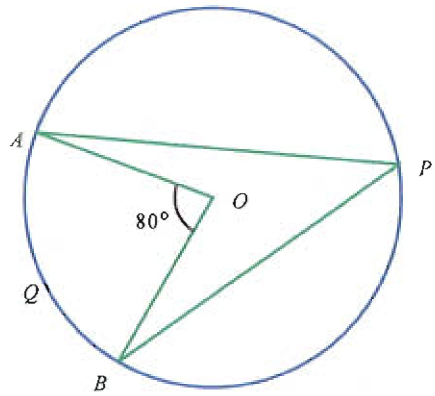
And it remains $130°$ till it reaches B.

> *Two points on a circle divide it into a pair of arcs. The angle got by joining these two points to a point on one of these arcs is equal to half the central angle of the alternate arc.*

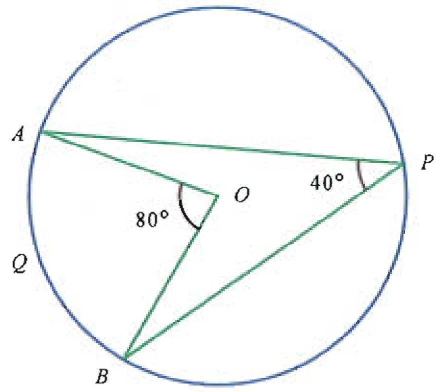
Here, instead of "central angle of an arc", we can say "angle made by an arc at the centre"; and instead of "angle got by joining ends of an arc to a point", we can say, "angle made by an arc at a point". Then the statement above can be rephrased like this:

> *The angle made by an arc at any point on the alternate arc is equal to half the angle made at the centre.*

For example, look at this picture:



The central angle of the arc AQB is 80°. So, $\angle APB$ made at the point P on the alternate arc is half of this, which is 40°.

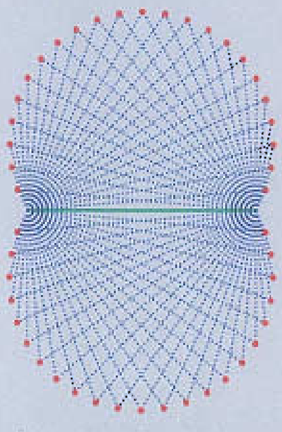


From the first picture, we can also see that the central angle of the arc APB is $360° - 80° = 280°$.

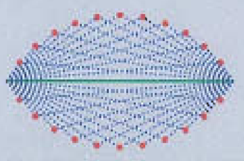
Circle trick

We started by drawing some pictures with the same angles above and below a line.

This was the picture got by taking 60° above and below:

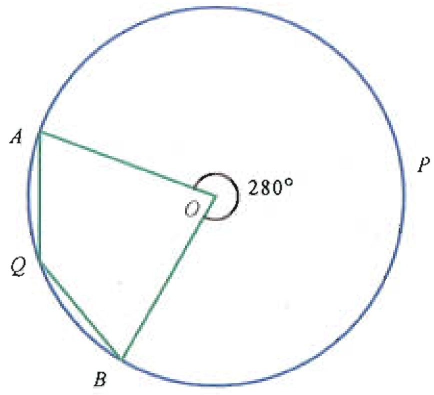


And taking 120° instead, we got this picture:



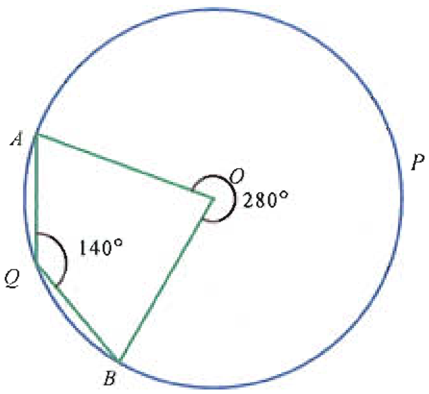
Now draw a picture with 60° above and 120° below. Don't you get a single, full circle? Why is this?

Suppose we take 30° above. What angle should we take below to get a full circle?



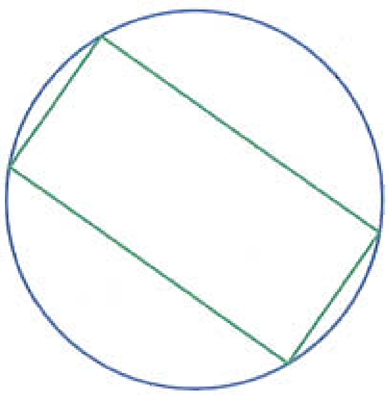
So, $\angle AQB$ made at the point Q on its alternate arc is

$\frac{1}{2} \times 280° = 140°$



Let's look at a few more examples:

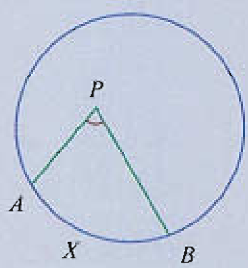
- In the figure below, all four vertices of the rectangle are on the circle. Prove that the diagonal of the rectangle is a diameter of the circle.



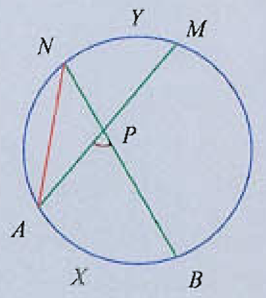
Inside the circle

What can we say about the angle got by joining the ends of an arc to a point inside the circle?

Look at this figure:



To find $\angle APB$, extend the lines AP and BP to meet the circle; join one of these points to one end of the arc:



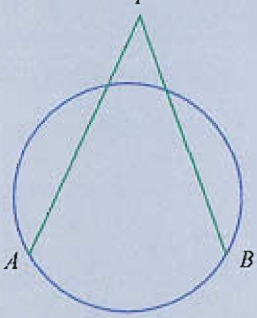
If we take the central angles of the arcs AXB and MYN as $x°$ and $y°$, then we get $\angle ANB = \frac{1}{2}\ x°$ and $\angle MAN = \frac{1}{2}\ y°$. These are angles of the triangle PAN; and $\angle APB$ is the exterior angle at the third vertex P. So,

$$\angle APB = \frac{1}{2}\ (x + y)°$$

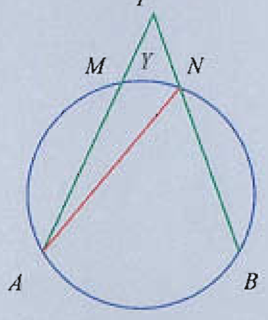
Thus $\angle APB$ is the mean of the central angles of the arcs AXB and MYN.

Outside a circle

What if we join the ends of an arc to a point outside the circle?



Join one end of the arc to the point where one of these lines cuts the circle.



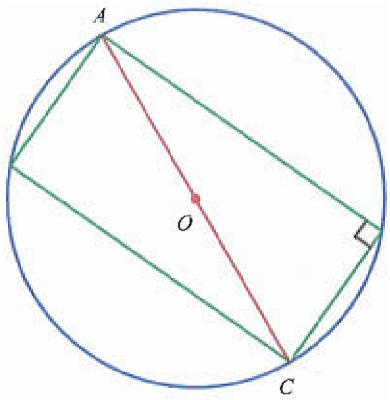
As before, let's take the central angles of the arcs AXB and MYN as $x°$ and $y°$. Again, we get $\angle ANB = \frac{1}{2}x°$ and $\angle MAN = \frac{1}{2}y°$. Here an external angle of ΔPAN is $\angle ANB$.

So,

$$\frac{1}{2}x = \angle APN + \frac{1}{2}y$$

from which we get

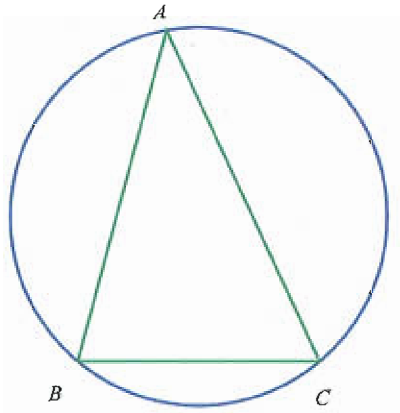
$$\angle APB = \frac{1}{2}(x - y)°$$



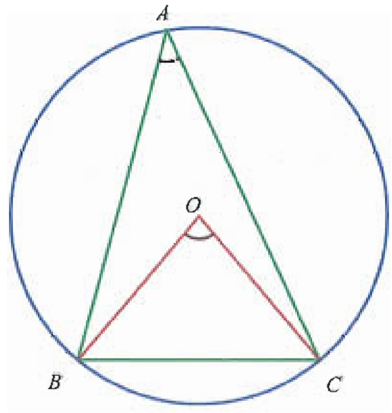
Since $ABCD$ is a rectangle, $\angle ADC = 90°$. So, the central angle of the alternate arc ABC of the arc ADC is $2 \times 90° = 180°$. That is, $\angle AOC = 180°$. This means that the points A, O, C lie on a straight line. In other words, AC is a diameter of the circle.

- How do we draw a triangle with angles 40°, 60°, 80° within a circle of radius 2.5 centimetres?

Let's first draw some triangle within a circle.



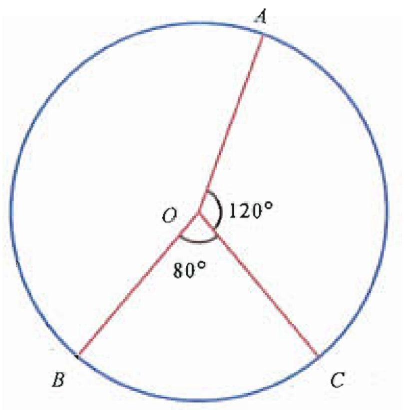
Join B and C to the centre O of the circle.



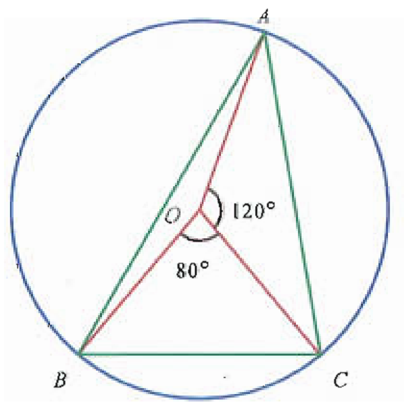
If $\angle BAC$ is to be 40°, how much should be $\angle BOC$?

Similarly, can't we find the angles got by joining other pairs of vertices to the centre of the circle?

So, for actually drawing the triangle we want, we draw a circle of radius 2.5 centimetres and mark *A, B, C* as shown below:

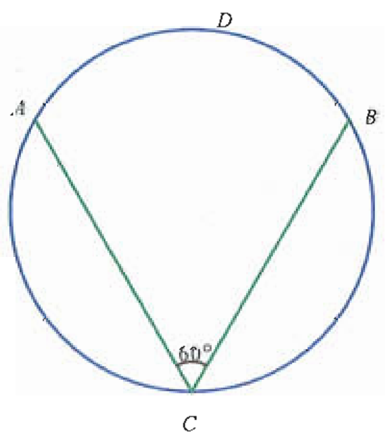


Now we need only join *A, B, C* to get the required triangle.



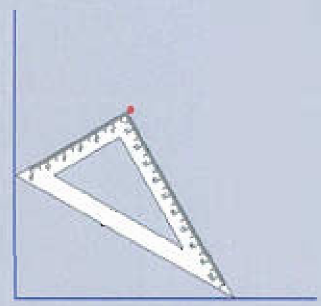
The problems below are for you:

- In this figure, what fraction of the circumference of the circle is the length of the arc *ADB*?

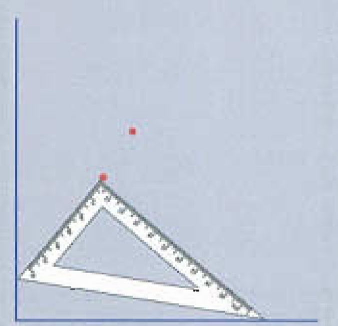


Sliding setsquare

Draw a pair of perpendicular lines on paper and place a set-square as shown below:



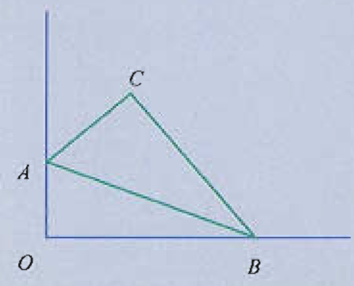
Mark the position of the top corner. Keeping the other two vertices on the lines, slide the set squre, and mark the various positions of the top vertex.



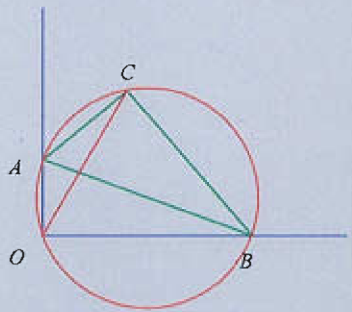
Note anything special about the points marked?

Circle, right angle and line

In our experiment with a sliding set square, all points marked are on a straight line, aren't they? Why does this happen?



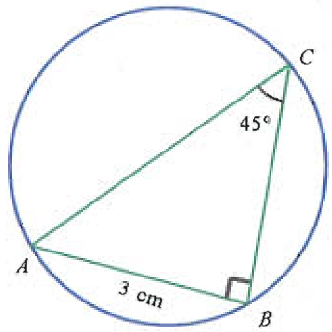
In the picture above, the set square is shown as ΔABC. Since both ACB and AOB are right angles, the circle on AB as diameter passes through both O and C.



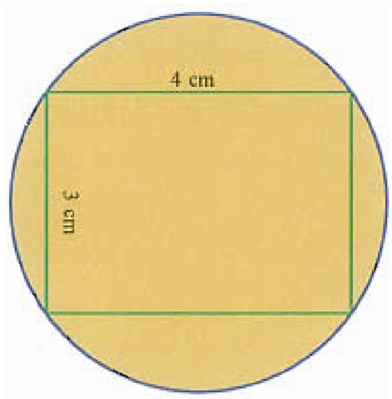
So, $\angle BAC = \angle BOC$. In this, since $\angle BAC$ is an angle of the set square we are using, it doesn't change. (In fact in the picture above, it is 60°.)

So, in sliding the set square, though the position of C changes, the line joining C and O keeps the same slant with OB. In other words, C can only move along the line making this angle with OB.

- What is the radius of the circle shown below?



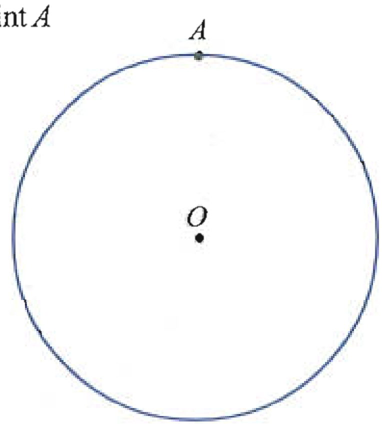
- What is the area of the circle shown below?



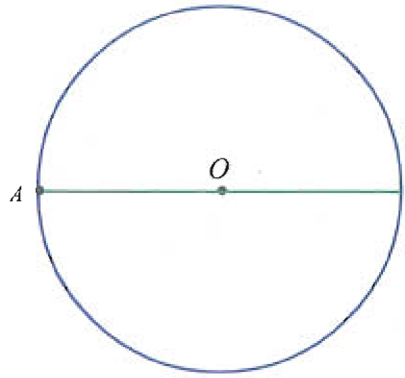
- How do we draw a triangle with two of the angles 40° and 120°, and circumradius 3 centimetres?

- How do we draw a $22\frac{1}{2}^{\circ}$ angle?

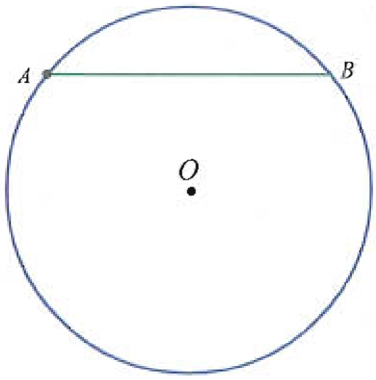
- In each of the pictures below, draw a $22\frac{1}{2}^{\circ}$ angle, according to the specifications:
  - At the point A



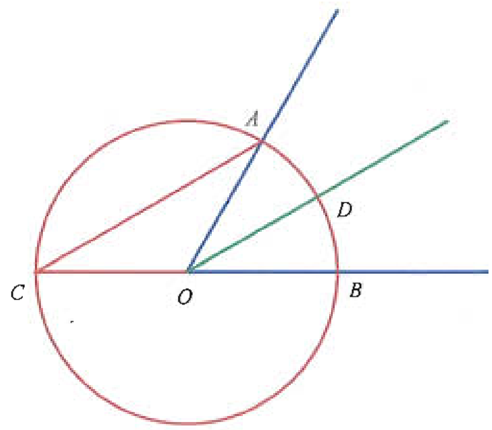
- At the point A with one side along OA



- At the point A, one side along AB



- In the picture below, O is the centre of the circle and the line OD is parallel to the line CA.

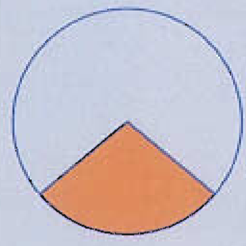


Prove that OD bisects $\angle AOB$.

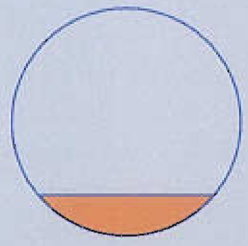
Can we use this to draw the bisector of a given angle? How?

Sector and segment

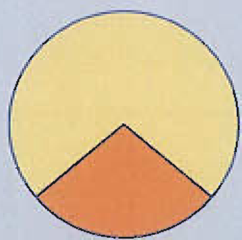
A sector of a circle is the part consisting of an arc of a circle and the two radii connecting its ends to the centre.

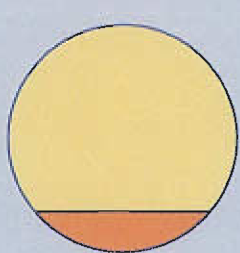


The part consisting of an arc and the chord joining its end points is a segment of a circle.

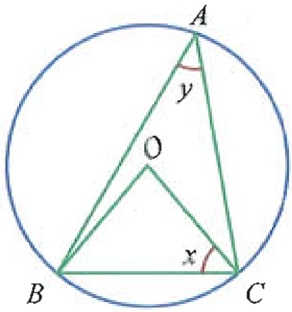


Since arcs of a circle come in pairs, so do sectors and segments





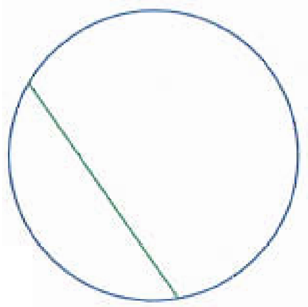
- In the figure below, O is the centre of the circle.

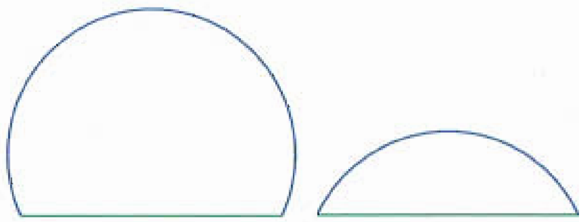


Prove that $x + y = 90°$

Segments of a circle

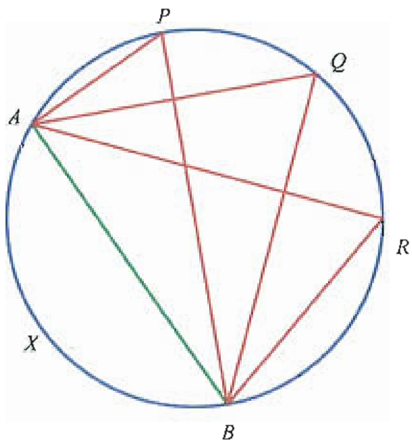
Every chord of a circle divides it into two parts:





Such a part is called a *segment* of a circle.

See this picture:

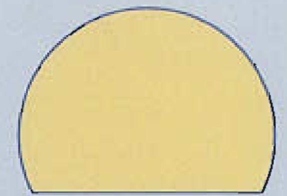


Each of $\angle APB$, $\angle AQB$, $\angle ARB$ is equal to half the central angle of the arc AXB and so they are all equal.

Size of a segment

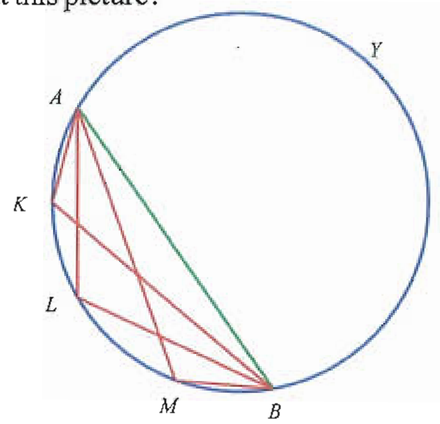
As the length of an arc increases, so does the size of the sector or segment made by it. And the length of an arc is measured using its central angle (Isn't it easier to measure the central angle than the length of the arc?)

The central angle of a sector is readily seen. What about the central angle of a segment?



First we have to determine the centre. And that we have already seen. (The section **Another view,** of the lesson **Circles** in the Class 9 textbook.)

What about this picture?



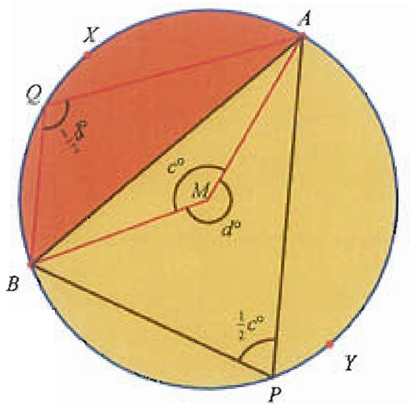
In this, each of $\angle AKB$, $\angle ALB$, $\angle AMB$ is equal to half the central angle of the arc AYB and so are all equal.

We can state this as follows:

Angles in the same segment of a circle are equal

We can note another fact also. Every segment has an alternate segment ; that is, a chord divides a circle into a pair of segments. Of these, angles in the same segment are equal. What about an angle in one segment and an angle in the alternate segment?

Angles in one segment are equal to half the central angle of an arc and the angles in the alternate segment are equal to half the central angle of the alternate arc, right? And what is the sum of the central angles of an arc and its alternate arc?



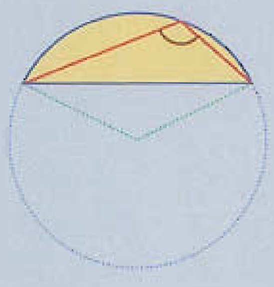
In the picture above, $c + d = 360°$ and so $\frac{1}{2}c + \frac{1}{2}d = 180°$.

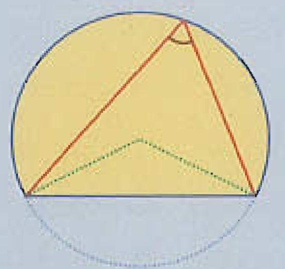
That is,

$$\angle APB + \angle AQB = 180°$$

Angle in a segment

Is there a direct method to compute the central angle of the arc of a segment, without finding the centre of the circle first? We know that angles in the same segment are equal.Using this angle, we can compute the central angle of the arc.





If the angle in a segment is $x°$, what is the central angle of its arc?

Circumcircle

We have seen that a circle can be drawn through any three points not on a line. (The section **Three points,** of the lesson **Circles** in the Class 9 textbook.) In other words, for any triangle, a circumcircle can be drawn.

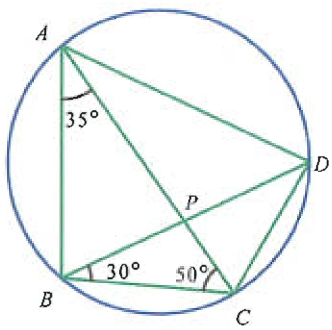
What about quadrilaterals? Rectangles and certain types of trapeziums have circumcircles; but parallelograms which are not rectangles do not have circumcircles. Thus among quadrilaterals, there are two classes: those with circumcircles and those without.

This can be stated as follows:

Angles in alternate segments are supplementary.

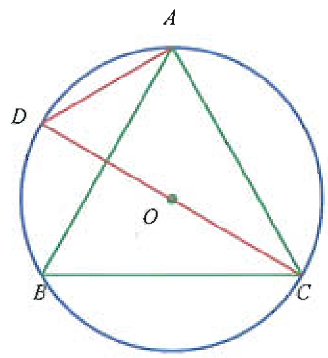
Now try your hand at these problems:

- In the figure below, A, B, C, D are points on the circle.



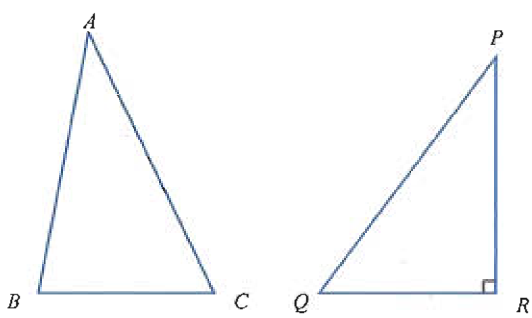
Compute the angles of the quadrilateral $ABCD$ and the angles between its diagonals.

- In the figure below, ΔABC is equilateral and O is its circumcentre.



Prove that the length of AD is equal to the radius of the circle.

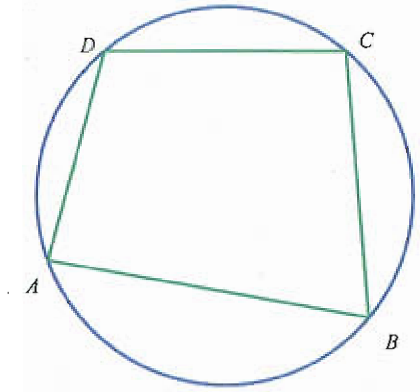
- In the picture below, ΔPQR is right angled. Also, $\angle A = \angle P$ and $BC = QR$.



Prove that the diameter of the circumcircle of ΔABC is equal to the length of PQ.

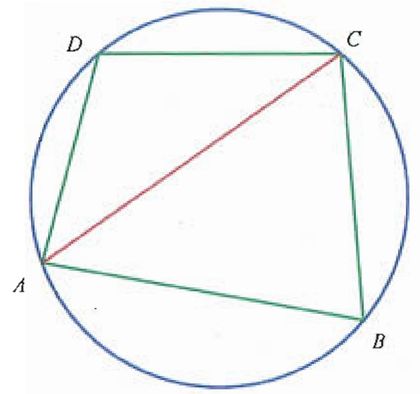
Circles and quadrilaterals

Look at this picture:



Is there any relation between the angles at *A, B, C, D*?

If you can't tell at once, try joining *AC*



Now the angles at *B* and *D* are the angles in the alternate segments which the chord *AC* cuts from the circle. And so they are supplementary.

Similarly, by drawing *BD*, we can see that the angles at *A* and *C* are also supplementary.

So, what can we say in general?

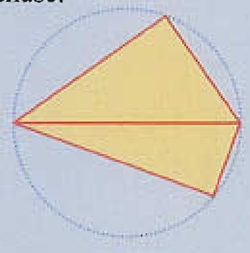
If all vertices of a quadrilateral are on a circle, then its opposite angles are supplementary.

Is the reverse true? That is, if the opposite angles of a quadrilateral are supplementary, can we draw a circle through all four of its vertices?

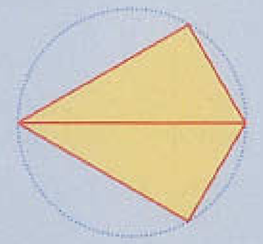
To answer this, let's first see how we would practically decide whether all four vertices of a quadrilateral can be put on a circle.

Making quadrilaterals

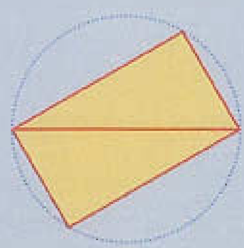
It is easy to construct special quadrilaterals which have circumcircles. One way is to join two right - angled triangles with equal hypotenuse:



What sort of a quadrilateral do we get if such triangles are actually congruent?



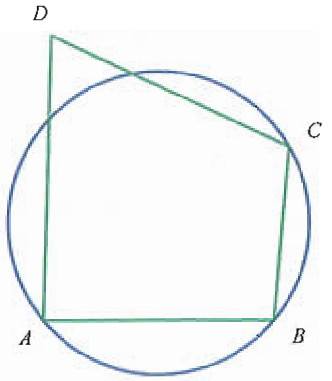
Suppose we flip the triangle below:



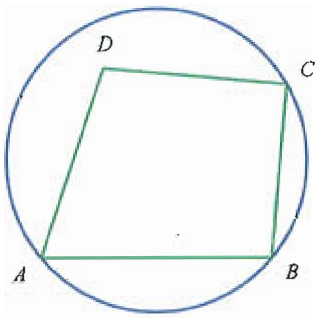
Can we make such quadrilaterals with circumcircles, using triangles other than right angled ones?

What should be the relation between the upper and lower triangles?

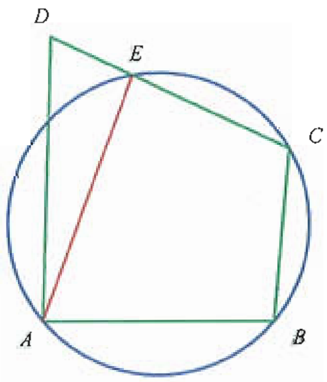
We can anyway draw a circle through three of the vertices of a quadrilateral. (Recall how we drew circles through three points not on a line, in Class 9.) Now the fourth vertex. If it is on this circle, then we are done. But this vertex may be outside the circle:



Or, inside the circle:



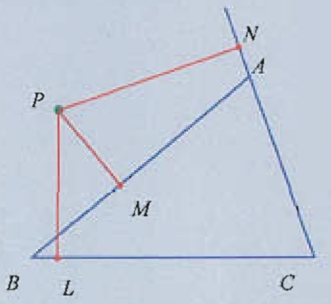
Let's have a closer look at the first figure. Joining A with the point where the circle cuts CD, we have another quadrilateral $ABCE$.



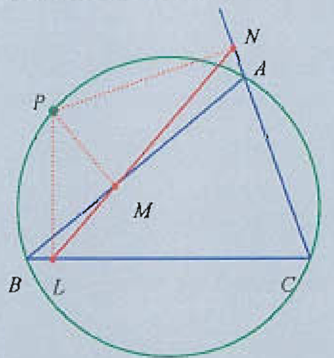
Since A, B, C, E are on the circle,

Circles and lines

We can check whether a specified point is on the circumcircle of a specified triangle, by measuring angles. There's another way. Draw perpendiculars from the point to the sides of the triangle:



If the feet of these perpendiculars lie on the same line, then the points are on the circumcircle; otherwise not.



This is known as Simpson's Theorem

(1) $$\angle B + \angle AEC = 180°$$

Now as in the discussion about points inside and outside the circle in the section **Right angles and circles**, we can see that

$$\angle AEC = \angle EAD + \angle D$$

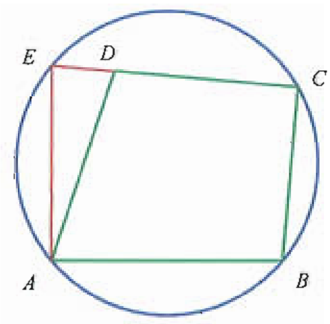
and so

(2) $$\angle D < \angle AEC$$

Thinking about the meanings of the algebraic statements marked (1) and (2) for a moment, we can easily see that

$$\angle B + \angle D < 180°$$

Next, in the second picture, we extend CD to meet the circle and join this point to A.



In this figure, we see that

(3) $$\angle B + \angle E = 180°$$

Also, from ΔAED, we find

$$\angle ADC = \angle E + \angle EAD$$

which gives

(4) $$\angle ADC > \angle E$$

From the statements marked (3) and (4), we get

$$\angle B + \angle ADC > 180°$$

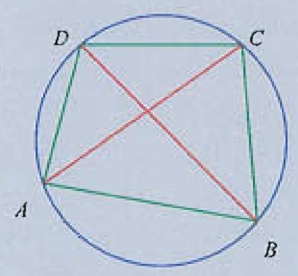
So, what have we seen?

Suppose a circle is drawn through three vertices of a quadrilateral. If the fourth vertex is outside this circle, then the sum of the angles at this vertex and the opposite vertex is less than 180°; if the fourth vertex is inside the circle, then this sum is greater than 180°.

Another theorem

Simpson's Theorem can also be seen as a theorem about quadrilaterals which have circumcircles. (How?) Another property of such quadrilaterals is that the sum of the products of opposite sides is equal to the product of the diagonals. That is, if the quadrilateral $ABCD$ has circumcircle, then

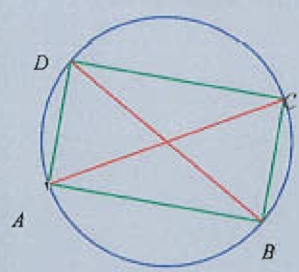
$$(AB \times CD) + (AD \times BC) = AC \times BD$$



On the other hand, if any quadrilateral satisfies this property, then it has a circumcircle. This is known as Ptolemy's Theorem.

A rectangle has a circumcircle. Also, its opposite sides are equal and so are the diagonals. So according to this theorem, in a rectangle $ABCD$

$$AB^2 + BC^2 = AC^2$$



But this is Pythagoras Theorem!

(We have already seen that if the fourth vertex is within this circle, then this sum is actually *equal* to 180°.)

Now suppose that in a quadrilateral $ABCD$, we have $\angle B + \angle D = 180°$. Draw the circle through A, B, C.

Can D be outside the circle? If so, we must have $\angle B + \angle D < 180°$. So, D is not outside the circle.

Can it be inside the circle? If so, we must have $\angle B + \angle D > 180°$. So, it is not inside the circle either.

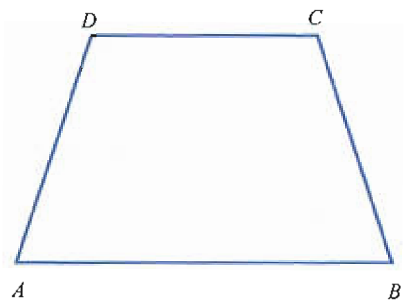
Since D is neither inside the circle nor outside, it must be on the circle.

That is,

> *If the opposite angles of a quadrilateral are supplementary, then we can draw a circle through all four of its vertices.*

A quadrilateral for which a circle can be drawn through all the four vertices is called a *cyclic quadrilateral*. From our discussion above, a cyclic quadrilateral can also be described as a quadrilateral in which the opposite angles are supplementary.

All rectangles are cyclic quadrilaterals. Isosceles trapeziums are also cyclic quadrilaterals. Look at this picture:



$ABCD$ is an isosceles trapezium. So,

$$\angle A = \angle B$$

(Recall the section **Isosceles trapeziums**, of the lesson **Construction of Quadrilaterals** in the Class 9 textbook.)

Also, since AB and CD are parallel,

$$\angle A + \angle D = 180°$$

Area

The Indian mathematician Brahmagupta has given a method to compute the area of a cyclic quadrilateral. If we take the lengths of the sides of a cyclic quadrilateral as a, b, c, d and $s = \frac{1}{2}(a + b + c + d)$, then its area is equal to

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Haven't we seen a similar formula? If we take $d = 0$ in this, it becomes

$$\sqrt{(s-a)(s-b)(s-c)s}$$

Isn't this Heron's formula for the area of a triangle?

Brahmagupta has also given methods to compute the areas of non - cyclic quadrilaterals, in terms of their sides and angles. Using these, we get another interesting property of cyclic quadrilaterals:

> *Among the various quadrilaterals with the same lengths for sides, the cyclic quadrilateral has the maximum area.*

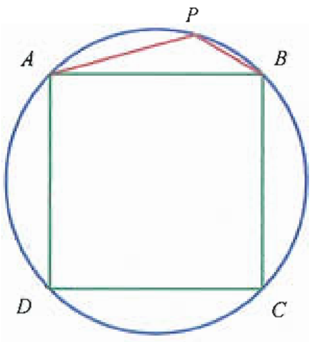
From these two equations, we get

$$\angle B + \angle D = 180°$$

This means $ABCD$ is a cyclic quadrilateral.

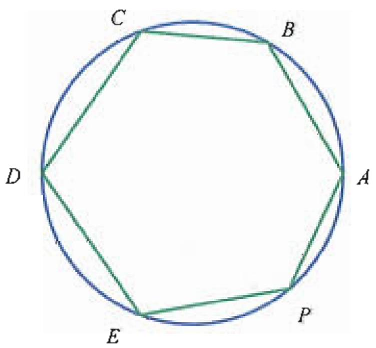
Now try these problems:

- Prove that in a cyclic quadrilateral, the exterior angle at any vertex is equal to the interior angle at the opposite vertex.
- Prove that a non-rectangular parallelogram is not a cyclic quadrilateral.
- Prove that non-isosceles trapeziums are not cyclic.
- In the figure below, $ABCD$ is a square.



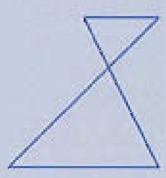
How much is $\angle APB$?

- Prove that in a cyclic hexagon $ABCDEF$ as shown below, $\angle A + \angle C + \angle E = \angle B + \angle D + \angle F$.

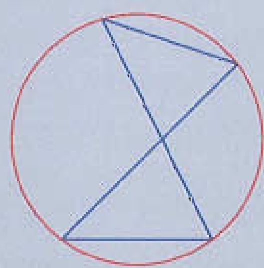


Similar triangles

We have seen several methods of drawing triangles similar to a given one. One of the methods is this:



After extending the sides upwards, instead of drawing a line parallel to the bottom side, draw a circle through the ends of the lines.

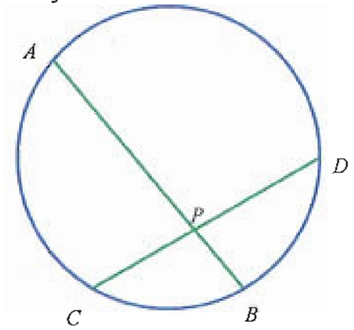


Is the upper triangle similar to the lower one?

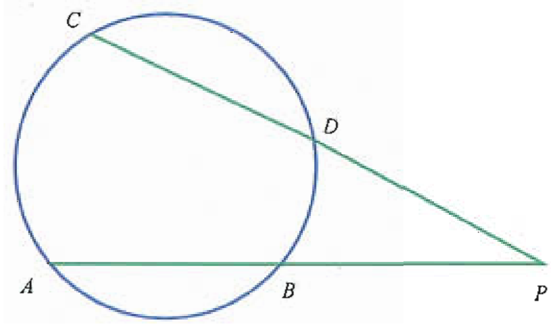
Cutting chords

Draw two non-parallel chords in a circle. They must intersect each other.

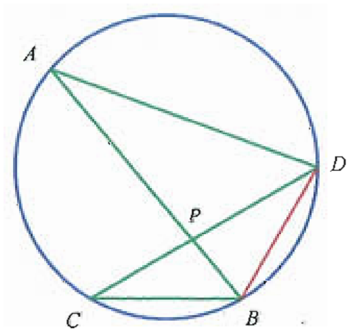
The intersection may be within the circle:

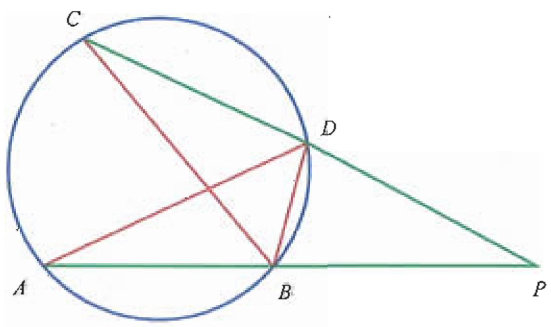


Or, outside the circle:



Either way, the triangles got by joining *AD* and *BC* can be shown to be similar.

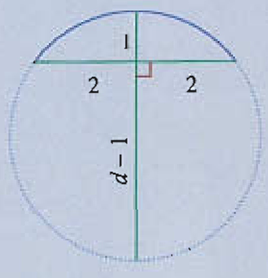




Old problem, new method

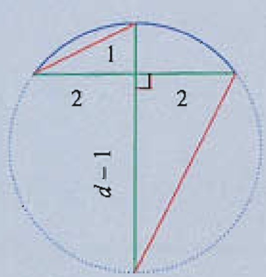
The distance between the ends of a piece of a bangle is 4 centimetres and its maximum height is 1 centimetre. Can we compute the radius of the bangle?

Remember doing this problem in Class 9? Now we can do it a little more quickly. We can picture the full bangle as below:



Here, d is the diameter of the circle.

We can draw two right angled triangles as below:



These are similar. (Why?) So,

$$\frac{d-1}{2} = \frac{2}{1}$$

which gives $d - 1 = 4$ and we get $d = 5$.

In both figures, look at the angles at A and C of ΔAPD and ΔBPC. They are the angles in the same segment cut off by the chord BD of the circle and so are equal.

In the first figure, the angles $\angle APD$ and $\angle BPC$ at P are the opposite angles formed by the intersection of the lines AB and CD and so are equal; in the second figure, they are just different names for the same angle.

Thus in either figure, two pairs of angles of ΔAPD and ΔBPC are equal and so the third pair is also equal. That is, the triangles are similar. Since in similar triangles, pairs of sides opposite equal angles have the same ratio, we have in the first figure

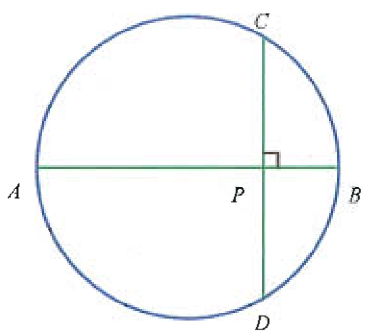
$$\frac{AP}{CP}=\frac{PD}{PB}$$

The second figure also gives the same. (Did you check?)

From this equation, we get

$$AP \times PB = CP \times PD$$

Let's look at a special case of this: AB is a diameter and CD is a chord perpendicular to it:



Since the perpendicular from the centre bisects a chord, we have $CP = PD$ here. So, the equation seen earlier becomes

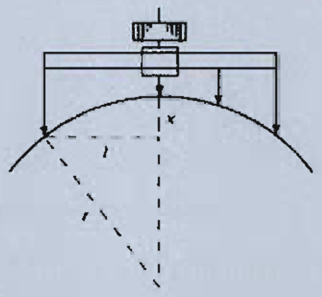
$$AP \times PB = CP^2$$

One application of this is to draw squares of any area. (One method of doing this is already seen in the section **Algebra and Pythagoras,** of the lesson **Irrational Numbers** in the Class 9 textbook.)

Math tool

Some lenses are made from spheres. And often we need to compute the radius of the sphere from which the lens was cut out. An instrument for this is the *spherometer.*

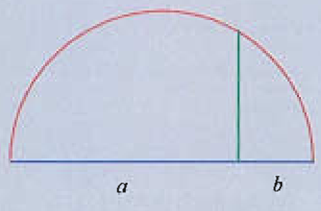
By placing its three legs on top of the piece of sphere, we can find the width of an arc. Using the screw in the middle, the maximum height can also be found.



With these, the radius can be computed as in our bangle problem.

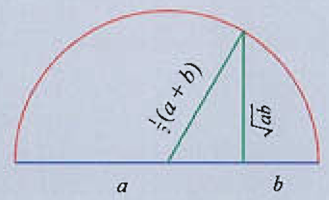
Geometry, algebra, numbers

Look at this picture:



What is the height of the vertical line? If we take it as x, we get $ab = x^2$ and so $x = \sqrt{ab}$

What is the radius of the semicircle? Since the diameter is $a + b$, the radius is $\frac{1}{2}(a+b)$.



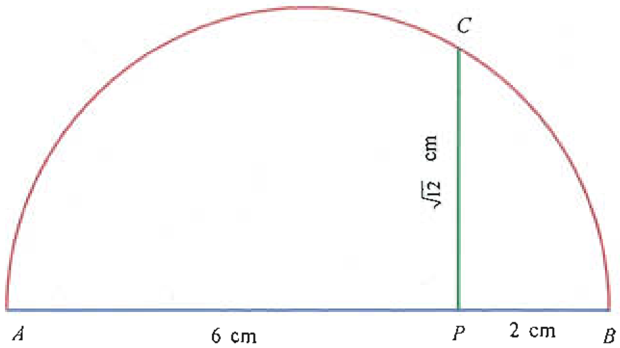
From the figure, we can see that the radius is larger than the perpendicular. Is there any instance in which they are equal? So, what do we see here?

For any two distinct numbers a and b, we have

$$\frac{1}{2}(a+b) > \sqrt{ab}$$

For example, let's see how we draw a square of area 12 square centimetres. What we need is a line with the square of its length 12 centimetres. In the equation above, the square of a length is expressed as the product of two other lengths. So, let's first write the square of the length we want, 12, as the product of two numbers. Since $12 = 6 \times 2$, if we draw $AP = 6$ and $PB = 2$ in the figure above, then by our equation, we will get $CP^2 = 12$.

So, we first draw AB of length 8 centimetres and mark P on it, 6 centimetres from A. Then we draw a semicircle with diameter AB. If we draw the perpendicular to AB through P and intersect the semicircle at C, we are almost done.



All it remains is to draw a square with CP as a side. (Remember the section **Square root** of the lesson **Real Numbers** in the Class 9 textbook?)

In how many different ways can we draw such square?

The problems below are for you:

- Draw a rectangle of sides 5 centimetres and 4 centimetres and draw a square of the same area.
- Draw a triangle of sides 4, 5, 6 centimetres and draw a square of the same area.
- Draw a quadrilateral with sides 2, 3, 4, 6 centimetres and one diagonal 5 centimetres. Draw a square of the same area.
- Given a quadrilateral, how do we draw a square of the same area, without making any measurements?