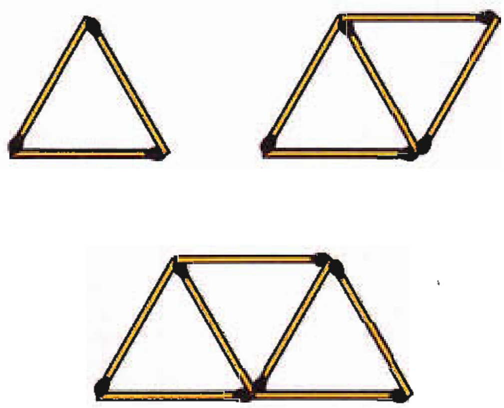
1 Arithmetic Sequences

Figure it out

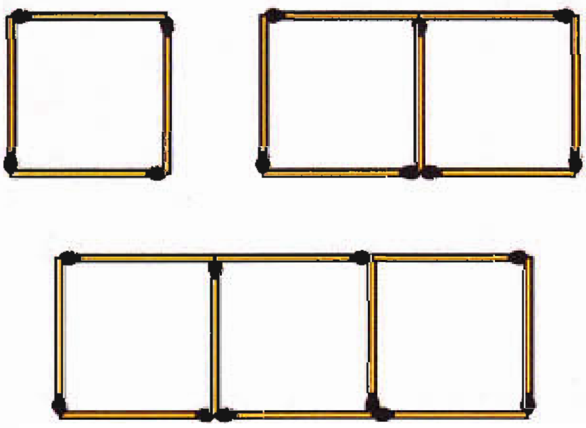
Look at these figures:



How many matchsticks are used in each?

How many matchsticks do we need to make the next figure in this pattern? (See the section **Matchstick math,** of the lesson **Letter Math** in the Class 6 textbook)

Now look at these:



How many sticks are used in each?

How many sticks do we need to make the next figure?

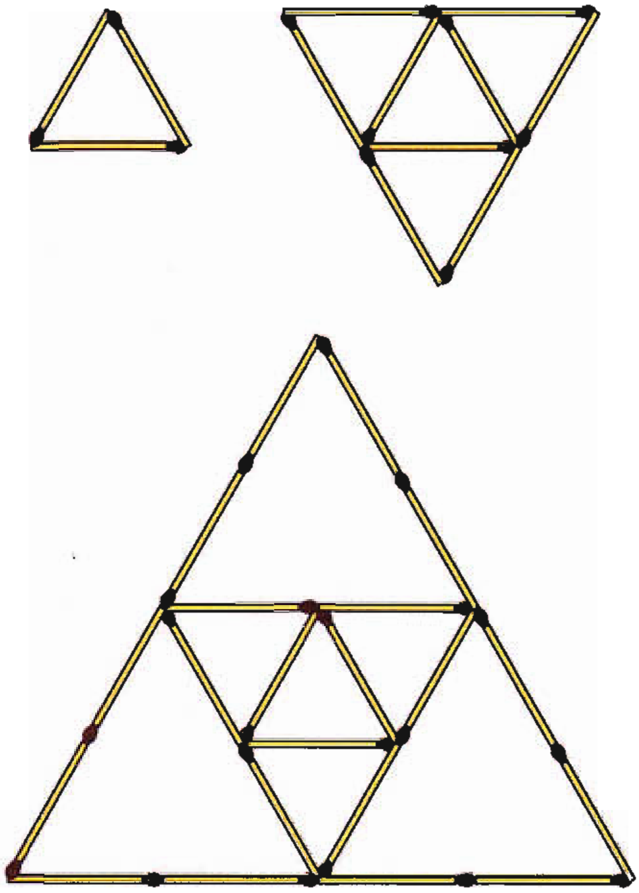
A game

This is a game for two. One of the players begins by saying aloud a number less than or equal to ten. The second player adds to this, a number less than or equal to ten. Then the first player makes it still larger by adding a number less than or equal to ten. The first to reach hundred wins.

For example, if the first player says 6, the second can make it 16, at the most. If he actually makes it 16, the first player can take it upto 26.

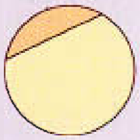
There is a scheme by which the first player can win. What are the numbers he should say to make sure that he wins? (Think backwards from 100)

Can you find out the number of matchsticks in each figure of the pattern below and also how many we need to make the next one?

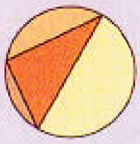


Circle division

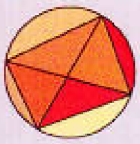
If we choose any two points on a circle and join them with a line, it divides the circle into two parts:



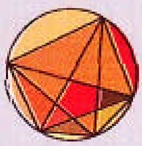
If we choose three points instead and join them with lines, they divide the circle into four parts:



What if choose four points and join every pair?



How about five points?



How many parts do you expect to get by joining six points? Check your guess by drawing a picture.

Numbers in order

Let's write down in order, the number of matchsticks used in each pattern above.

In the first pattern of triangles, we have

3, 5, 7, 9, ...

In the second pattern of squares,

4, 7, 10, 13, ...

What about the last pattern of triangles?

$$3, 9, 21, 45, \ldots$$

A collection of numbers as in these examples, written in order as the first, second, third and so on according to a definite rule, is called a *number sequence*.

Let's look at some more instances of such sequences:

- Suppose we deposit 1000 rupees in a bank and that we get simple interest at 6% annually. What would be the amount at the beginning of each year?

  At the beginning of the first year, it is 1000 rupees. At the beginning of the second year, it becomes 1060 rupees. At the beginning of the third year, it becomes 1120 rupees. And this goes on, right?

  That is, we get the sequence

  $$1000, 1060, 1120, 1180, \ldots$$

  What if the interest is compounded annually?

  We get the sequence

  $$1000, 1060, 1124, 1191, \ldots$$

  instead.

- We know that the speed of an object falling towards the earth increases every second by 9.8m/s. In other words, for an object dropped from a height, the speed is 9.8 m/s after the first second, 19.6 m/s after the second second, 29.4 m/s after the third second and so on. Thus the sequence of numbers we get here is

  $$9.8, 19.6, 29.4, \ldots$$

  What about the total distance this object travels after each second?

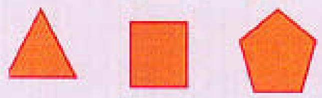
  It changes according to the equation, $s = 4.9t^2$. So, the total distance in metres that this object travels in one second, two seconds, three seconds and so on gives the sequence

  $$4.9, 19.6, 44.1, \ldots$$

Different kinds of sequences

The word sequence in ordinary language means things occurring one after another, in a definite order. In mathematics, we use this word to denote mathematical objects placed as the first, second, third and so on. The objects thus ordered need not always be numbers.

For example, we can have a sequence of polygons as given below:



Or a sequence of polynomials as

$$1 + x, 1 + x^2, 1 + x^3, \ldots$$

The arrangements of words of a language in alphabetical order is also a sequence.

Number sequences

Number sequences arise in any number of ways. For example, the digits in the decimal representation of π in order, gives the sequence

$$3, 1, 4, 1, 5, 9, \ldots$$

There is no simple way to find the number occuring at a specific position in this sequence.

In some sequences, the numbers repeat. For example, the digits in the decimal representation of $\frac{10}{11}$, written in order are

$$0, 9, 0, 9, \ldots$$

If we write out in order the last digits of the powers 2, 2^2, 2^3, ...we get the sequence

$$2, 4, 8, 6, 2, 4, 8, 6, \ldots$$

- A tank contains 1000 litres of water and it flows out at the rate of 5 litres per minute. So after one minute, the tank would contain 995 litres, after two minutes, 990 litres and so on. Here, we get the sequence

$$1000, 995, 990, 985, \ldots$$

Now write down the number sequences got in the instances described below:

- The perimeters of squares with each side of length 1 centimetre, 2 centimetres, 3 centimetres and so on. Also, the areas of these squares.
- The sum of the interior angles of polygons of sides 3, 4, 5 and so on. And the sum of their exterior angles.
- Multiples of 3

  Numbers which leave a remainder 1 on dividing by 3

  Numbers which leave a remainder 2 on dividing by 3
- Natural numbers which end in 1 or 6, written in order. Can you describe this sequence in any other way?

Add and go forth

We have seen so many sequences now. Let's have a look at them all together:

- 3, 5, 7, 9, ...
- 4, 7, 10, 13, ...
- 3, 9, 21, 45, ...
- 1000, 1060, 1120, 1180, ...
- 1000, 1060, 1124, 1191, ...
- 9.8, 19.6, 29.4, 39.2, ...
- 4.9, 19.6, 44.1, 78.4, ...
- 1000, 995, 990, 985, ...
- 4, 8, 12, 16, ...
- 1, 4, 9, 16, ...

- 180, 360, 540, 720, ...
- 360, 360, 360, 360, ...
- 3, 6, 9, 12, ...
- 1, 4, 7, 10, ...
- 2, 5, 8, 11, ...
- 1, 6, 11, 16, ...

The sequence 3, 5, 7, 9, . . . is got from the first matchstick problem. Here we need 3 sticks to make the first triangle and to make each new triangle, we need 2 more sticks.Thus we add 2 to 3, again add 2 to this sum and so on, to get the numbers in the sequence 3, 5, 7, 9, . . .

What about the second sequence? When we make squares with matchsticks, we need 4 sticks for the first square and for each new square, we need 3 more sticks. Thus we start with 4 and add 3 repeatedly to get the sequence 4, 7, 10, 13, . . .

Now look at the next sequence in our list. We need 3 sticks for the first triangle. To make a triangle on each of its sides, we need $3 \times 2 = 6$ more sticks; $3 + 6 = 9$ sticks in all. Next to make a triangle on each side of this bigger triangle, we need $3 \times 4 = 12$ more sticks; which means a total of $9 + 12 = 21$. Thus in this sequence, we start with 3 and continue by first adding 6, then 12, then 24 and so on.

> *A sequence which starts with any number and proceeds by the addition of one number again and again, is called an arithmetic sequence or an arithmetic progression.*

So, the first two sequences in our list are arithmetic sequences; the third is not.

Now look at the sequence 1000, 995, 990, . . . Here, the numbers progressively decrease by 5.

Subtracting 5 can also be described as adding –5, right? So, we can say that the numbers in this sequence are got by adding –5 again and again. Thus it is also an arithmetic sequence.

What about the sequence 360, 360, 360, . . . , got as the sum of the exterior angles of polygons of increasing number of sides? Here we can say that each number is got by adding 0 repeatedly to 360. So, it is also an arithmetic sequence.

Natural number sequences

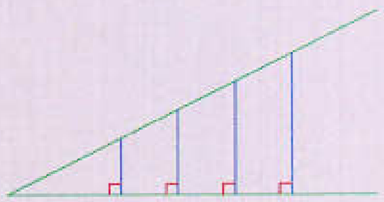
Some mathematicians have got together to collect sequences of natural numbers arising in various contexts and put them in the Net. The result is The Online Encyclopedia of Integer Sequences available at http://oeisf.org. It contains about 175,000 sequences.

At the web-page www.research.att.com/~njas/sequences/index.html, we can enter a few natural numbers and get several sequences containing these numbers in the same order, and short descriptions of how these sequences arise.

For example, entering 1, 2, 3, 4, 5, 7, we get 455 different sequences containing these numbers in this order. Some of them are:

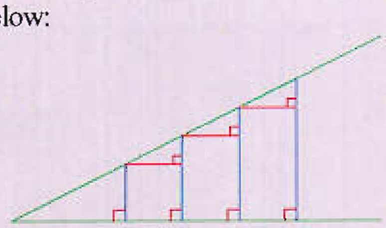
- 1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, . . . Powers of primes in ascending order
- 1, 2, 3, 4, 5, 7, 8, 9, 10, 12,14, . . . Natural numbers which give a prime on multiplying by 6 and subtracting 1
- 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, . . . Numbers which do not have two consecutive natural numbers (other than 1) as factors

Parallel progression



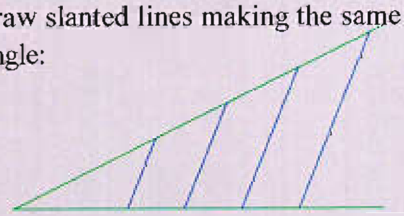
The vertical lines in the picture are equally spaced. Can you prove that their heights are in arithmetic sequence?

Draw perpendiculars as in the picture below:



The small right angled triangles got thus are all congruent. (Why?) So, their vertical sides are equal. That is, in the first picture, the heights of adjacent vertical lines increase by the same amount. This means the heights of these lines are in arithmetic sequence.

Suppose instead of vertical lines, we draw slanted lines making the same angle:



Are their lengths also in arithmetic sequence?

Now check which of the sequences in our list above are arithmetic sequences. Then have a look at these problems:

- The multiples of 2, written in order give 2, 4, 6, 8, . . . Is this an arithmetic sequence? What about the powers 2, 4, 8, 16, . . . of 2?
- Dividing the natural numbers by 2, we get $\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ Is this an arithmetic sequence?
- In the arithmetic sequence got by adding $\frac{1}{4}$ to itself repeatedly, does the number 10 occur anywhere? What about 11?
- The reciprocals of the natural numbers in order give $1, \frac{1}{2}, \frac{1}{3}, \ldots$ Is it an arithmetic sequence?
- Write down the sequence of differences (subtracting the larger from the smaller) of consecutive perfect squares. Is it an arithmetic sequence?

Addition and subtraction

Look at the two sequences got from the problems on interest. In both sequences, the increase in numbers is due to the interest added. In the case of simple interest, the increase every year is the interest for 1000 rupees (that is, 60 rupees). So, even without any computation, we can see that the total amounts form an arithmetic sequence.

What about compound interest? The interest added changes every year; so the total amounts are not in arithmetic progression.

What about the speed problem? What number should we add to 9.8 to get 19.6?

$$19.6 - 9.8 = 9.8$$

And what number do we need to add to 19.6 to make it 29.4?

$$29.4 - 19.6 = 9.8$$

That is, the sequence progresses by the addition of 9.8 itself at every stage. So speeds are in arithmetic sequence.

What about distances?

What number added to 4.9 gives 19.6?

$$19.6 - 4.9 = 14.7$$

And what number added to 19.6 gives 44.1?

$$44.1 - 19.6 = 24.5$$

So, the sequence of distances is got by starting with 4.9, adding 14.7 first, then adding 24.5, ...

So, it is not an arithmetic sequence. (Why?)

Do you notice something else in these examples?

In an arithmetic sequence, if we subtract any number from the number just after it, we get the same number

This number is called the *common difference* of the arithmetic sequence. In other words, common difference is the number repeatedly added to get the numbers in an arithmetic sequence.

As an example, look at the three sequences we got, based on division by 3:

3, 6, 9, 12, ...

1, 4, 7, 10, ...

2, 5, 8, 11, ...

What is the common difference of each sequence?

Let's look at some problems:

- The first number of an arithmetic sequence is 10 and the third number is 24. What is the second number?

  According to the facts given, by adding a number to 10 and then adding the same number again, we get 24 (How is that?)

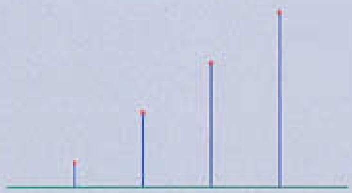
  So, double this number is $24 - 10 = 14$. This means the number added is 7.

  This gives the second number as $10 + 7 = 17$

  There's is another way to do this. We get the same number if we subtract the first number from the second, or if we subtract the second number from the third. (Look up the meaning of common difference.)

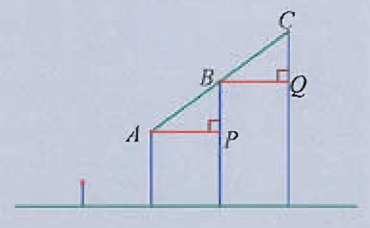
Geometry of arithmetic sequence

Take an arithmetic sequence of positive numbers. Draw a line and draw equally spaced perpendiculars to it, with heights equal to the numbers in this arithmetic sequence.



Join the tops of these perpendiculars. Aren't they all on the same line? Why is this so?

Take any three adjacent vertical lines and join their tops. Also draw perpendiculars as below:



The triangles ABP and BCQ are congruent. (Why?) And this means their angles are equal.

So, if we take $\angle ABP = x°$, then

$$\angle ABC = x + 90 + (90 - x) = 180°$$

This means A, B, C are all on the same straight line.

Back and forth

We have seen that the sum of three consecutive natural numbers is thrice the middle number. (The section, **Clarity in expression** of the lesson, **Polynomials** in the Class 9 textbook.)

This can be seen in another manner. The first number is 1 less than the middle number; and the third number is 1 more. So, when we add all three, the added 1 and the subtracted 1 cancel out, leaving only the middle number repeated thrice.

Translating this into algebra, if we take the middle number as x, then the left number is $x - 1$ and the right number is $x + 1$. And the sum is

$$(x-1)+x+(x+1)=3x$$

Instead of consecutive natural numbers, suppose we take any three consecutive numbers from an arithmetic sequence. The decrease and increase from the middle number would be the common difference, instead of 1. And we get the same result.

How about translating this reasoning into algebra? Let's take the common difference as d. If we take the middle number as x, then the numbers are $x-d, x, x+d$ and so the sum is

$$(x-d)+x+(x+d)=3x$$

Suppose instead of three, we take five numbers? Seven?

So, if we take the second number as x, then we get

$$x - 10 = 24 - x$$

From this, we get

$$2x = 24 + 10 = 34$$

and this gives

$$x = 17$$

- Prove that in any arithmetic sequence, if we take three consecutive numbers, then the middle number is half the sum of the first and the last.

Let's take a, b, c as three consecutive numbers in an arithmetic sequence.

Then, as in the second method of the previous problem, we get

$$b - a = c - b$$

From this, we get

$$2b = a + c$$

and then

$$b = \frac{1}{2}(a + c)$$

Now try the problems below on your own:

- In each of the arithmetic sequences below, some numbers are missing. Their positions are marked with a ◯. Find these numbers.
  - 24, 42, ◯, ◯, ...
  - ◯, 24, 42, ◯, ...
  - ◯, ◯, 24, 42, ...
  - 24, ◯, 42, ◯, ...
  - ◯, 24, ◯, 42, ...
  - 24, ◯, ◯, 42, ...
- The second and fourth numbers in an arithmetic sequence are 8 and 2. Find the first and the third numbers.

- The second number of an arithmetic sequence is 5. Find the sum of its first and third numbers.
- Find an algebraic expression to compute the third number of an arithmetic sequence, using the first and the second numbers.

Position and term

In every sequence, the numbers are ordered as the first, second and so on. They are generally called *terms* of the sequence.

For example, in our first sequence got from triangles, the first term is 3, the second term is 5, the third term is 7 and so on.

What is its tenth term?

In other words, how many matchsticks do we need to make 10 triangles in this pattern? For this, we will have to join 9 more triangles with the first. For the first triangle we need 3 matchsticks and for every new triangle thereafter, we need 2 matchsticks. So,

$$\text{number of matchsticks for 10 triangles} = 3 + (9 \times 2) = 21$$

In other words, the 10^{th} term of the arithmetic sequence 3, 5, 7, . . . is 21.

Like this, can you compute the 15^{th} term of our second sequence 4, 7, 10, . . . got from squares?

Let's look at a few more problems:

- The first term of an arithmetic sequence is 2 and the common difference is 5. What is its 13^{th} term?

  Common difference is the number we add to each term of a sequence to go to the next term. Here it is 5; and the first term is 2. So, the terms of the sequence proceed as 2, 7, 12, ...

  How many steps are needed to go from the 1^{st} term to the 13^{th} term?

  That is, how many times do we need to add 5 to 2?

  So,

  $$13^{th}\text{ term} = 2 + (12 \times 5) = 62$$

Sum and mean

We saw that in an arithmetic sequence, the sum of three consecutive numbers is three times the middle number, the sum of five consecutive terms is five times the middle number and so on.

What if we take an even number of terms? From one viewpoint, there is no middle term; from another viewpoint, there is a pair of middle terms. (In 1, 2, 3, 4, 5, 6 can't we say that the pair 3, 4 is right in the middle?)

So, in an arithmetic sequence of common dfference d, if we take four consecutive terms and name the middle pair x, y, then these four terms are $x - d, x, y, y + d$. And their sum is $2(x + y)$, which we can write $4 \times \frac{1}{2}(x + y)$. This we can say in ordinary language as four times the mean of the middle pair. (Don't you remember the lesson **Statistics,** in the Class 9 textbook?)

Is this true for six consecutive terms? What about eight?

Proportional differences

In an arithmetic sequence, the difference of two consecutive terms is the common difference. What if we skip a term? The difference is twice the common difference, right?What if we skip two terms?

In an arithmetic sequence, the difference between any two terms is got by multiplying the difference of their positions by the common difference.

This can be stated in another manner: in any arithmetic sequence, the term difference is proportional to the position difference; and the constant of proportionality is the common difference.

- The 12^{th} term of an arithmetic sequence is 25 and the common difference is 3. What is the 17^{th} term?

To get the 17^{th} term from the 12^{th} term, how many times should we add the common difference?

$$17^{th} \text{ term} = 25 + (5 \times 3) = 40$$

- In an arithmetic sequence, the 5^{th} term is 32 and the 11^{th} term is 74. What are the numbers in this sequence?

To get the 11^{th} term from the 5^{th} term, we must add the common difference 6 times.

From the given facts, the number added is $74 - 32 = 42$

So, common difference is $42 \div 6 = 7$

We get the 5^{th} term by adding the common difference 4 times to the 1^{st} term. So, in this sequence, the 5^{th} term 32 is got by adding $4 \times 7 = 28$ to the 1^{st} term. So,

$$\text{first term} = 32 - 28 = 4$$

Since the first term is 4 and the common difference is 7, the terms are

$$4, 11, 18, \ldots$$

We can do this problem using algebra also. Taking the first term as x and the common difference as y, the given facts translate to the two equations

$$x + 4y = 32$$

$$x + 10y = 74$$

(How?) Solving them, we get $x = 4, y = 7$. (Recall the lesson **Pairs of Equations,** in the Class 9 textbook.)

- In the arithmetic sequence 3, 7, 11, . . . , is 101 a term? What about 103?

The terms start with 3 and proceed by repeatedly adding 4; that is, by adding to 3, the multiples 4, 8, 12, . . . of 4.

In other words, from any term of this sequence if we subtract 3, then we get a multiple of 4. And all such numbers are in this sequence.

Now let's look at 101 and 103

$$101 - 3 = 98$$

since 98 is not a multiple of 4, the number 101 is not a term of this sequence.

$$103 - 3 = 100$$

Since 100 is a multiple of 4, the number 103 is a term of this sequence.

Now do the problems given below on your own:

- The first term of an arithmetic sequence is 7 and the common difference is –2. What is its 12th term?
- The 3rd term of an arithmetic sequence is 10 and its 8th term is 25. What is its 4th term? And the 13th term?
- The 5th term of an arithmetic sequence is 11 and its 12th term is 32. What are its first three terms?
- The 5th term of an arithmetic sequence is 9 and its 9th term is 5. What is its common difference? What is the 14th term?
- The common difference of an arithmetic sequence is –1 and its 4th term is 7. What is its 7th term? And the 11th term?
- How many three digit numbers leave a remainder 3 on division by 4?
- Write down the sequence of natural numbers which leave a remainder 3 on division by 6. What is the 10th term of this sequence? How many terms of this sequence are between 100 and 400?

Sequences in Algebra

The terms of a sequence are formed by some rule. For example, in our first triangle problem, we start with 3 and repeatedly add 2 to get the sequence

$$3, 5, 7, \ldots$$

Sequence and remainders

The even numbers 2, 4, 6, . . . form an arithmetic sequence. So do the odd numbers 1, 3, 5, ... Both these have common difference 2.

Even numbers mean numbers divisible by 2; that is, numbers which leave a remainder 0 on division by 2. And odd numbers are those which leave a remainder 1 on division by 2.

Similarly, we have seen that there are three arithmetic sequences based on division by 3; those which leave a remainder 0, 1 or 2. What is the common difference of all these?

Based on the remainders got on division by 4, how many arithmetic sequences of natural numbers do we get? What are they? What is the common difference of all these?

Now let's think in reverse. In any arithmetic sequence of natural numbers, the difference of any two terms is a multiple of the common difference. This means they leave the same remainder on division by the common differnece. (Why?)

Thus, any arithmetic sequence of natural numbers is of the form described first; it consists of numbers leaving the same remainder on division by a specific number. And this divisor is the common difference.

Rules for sequences

What is the next term of the sequence 3, 5, 7, . . . ?

We haven't said that it should be an arithmetic sequence. So, the next number need not be 9. For example, if we had meant this to be the sequence of odd primes, then the next number would be 11.

What is the moral? From a few numbers written in order, we cannot predict the numbers to follow in the sequence. For this, the rule of forming the sequence or the context in which it arises should be specified.

Indeed, we have seen how the sequence 1, 2, 3, 4, 5, 7 can be continued in various ways, according to different rules, in the section **Natural number sequences.**

In general, to get the term at any specified position of this sequence, we subtract 1 from the position number, multiply this by 2 and add to 3. For example,

$$15^{th}\text{ term} = \big((15-1)\times 2\big) + 3 = 31$$

How about writing the general rule in algebra?

For every natural number n,

$$n^{th}\text{ term} = \big((n-1)\times 2\big) + 3 = 2n+1$$

So, in the algebraic expression $2n+1$, if we take $n = 1, 2, 3, \ldots$, we get the terms 3, 5, 7, . . . of this sequence in the correct order. (See the section, **Algebraic expressions** of the lesson, **Polynomials** in the Class 9 textbook.)

In algebraic discussions, we denote the terms of a sequence by symbols such as $x_1, x_2, x_3, \ldots$ or $y_1, y_2, y_3, \ldots$ Thus we can write the sequence in the example above as

$$x_1 = 3$$

$$x_2 = 5$$

$$x_3 = 7$$

....

....

We can shorten this further by writing

$$x_n = 2n+1, \text{ for every natural number } n$$

(Note that this contains all the information about this sequence.)

Let's write the sequence of the square problem in this shortened form. In this, the first number is 4 and the number repeatedly added is 3. So, for evey natural number n,

$$n^{th}\text{ term} = \big((n-1)\times 3\big) + 4 = 3n+1$$

Again, we can shorten this and compress the sequence into the algebraic expression

$$x_n = 3n+1$$

What about the sequence in the second triangle problem?

We can write the terms of this sequence as

$$x_1 = 3$$

$$x_2 = 9 = 3 \times (2^2 - 1)$$

$$x_3 = 21 = 3 \times (2^3 - 1)$$

$$x_4 = 45 = 3 \times (2^4 - 1)$$

and so on. In general, we can write the terms as

$$x_n = 3(2^n - 1)$$

(see the section **Growing triangle.**)

In the simple-interest problem, we got the arithmetic sequence 1000, 1060, 1120, ... The n^{th} term is

$$1000 + 60\,(n - 1) = 60n + 940$$

To get the sequence 1000, 1060, 1124, 1191, . . . of the compound interest problem, we have to first find the terms of the sequence given by

$$x_n = 1000\,(1.06)^{n-1}$$

and then round each number to the nearest natural number. (See the section **A computational trick,** of the lesson **Financial Math** in the Class 8 textbook.)

Now find the algebraic expression for all the sequences given at the beginning of the section, **Add and go forth**.

Algebra of arithmetic sequences

Look at the algebraic form of some of the arithmetic sequences we have seen:

3, 5, 7, 9, ...	$x_n = 2n + 1$
9.8, 19.6, 29.4, 39.2, ...	$x_n = 9.8n$
1000, 995, 990, 985, ...	$x_n = -5n + 1005$
4, 8, 12, 16, ...	$x_n = 4n$
360, 360, 360, 360, ...	$x_n = 360$
1, 4, 7, 10, ...	$x_n = 3n - 2$

In all these, the n^{th} term x_n is got by multiplying n by a specific number and then adding a specific number.

Growing triangle

We got the sequence 3, 9, 21, 45, . . . in our problem of making larger and larger triangles with matchsticks, How do we compute its terms?

In the original problem, we start with a triangle formed with one matchstick on each side; the construction progresses by constructing a larger triangle with two sticks on each side, then a still larger one with four sticks on each side and so on.

So, the total number of matchsticks needed at each stage is given by

$$3,\ 3 + (3 \times 2),\ 3 + (3 \times 2) + (3 \times 2^2), ...$$

That is,

$$3,\ 3(1 + 2),\ 3(1 + 2 + 2^2), ...$$

In this we can write

$$\begin{aligned} 1 + 2 &= 3 \\ &= 2^2 - 1 \\ 1 + 2 + 2^2 &= (2^2 - 1) + 2^2 \\ &= (2 \times 2^2) - 1 \\ &= 2^3 - 1 \\ 1 + 2 + 2^2 + 2^3 &= (2^3 - 1) + 2^3 \\ &= (2 \times 2^3) - 1 \\ &= 2^4 - 1 \end{aligned}$$

and so on. From this, we can compute the n^{th} term of our sequence as

$$3(1 + 2 + 2^2 + ... + 2^{n-1}) = 3(2^n - 1)$$

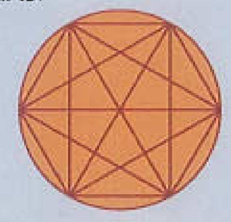
So, to make 25 triangles in this manner, we need $3(2^{25} - 1) =$ 100663293 matchsticks–more than ten crore!

Hasty conclusions

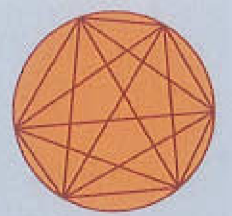
We discussed a problem on the number of parts of a circle got by joining points on it, in the section **Circle division**. For the number of points equal to 2, 3, 4, 5 we get the number of parts as 2, 4, 8, 16.

What about 6 points? We tend to guess the number of parts as 32. Do we get this many parts if we actually draw such a picture?

If the points are equally spaced, we get 30 parts:



Otherwise, 31 parts



Either way, the maximum number of parts is 31.

It can be proved that in general, if we take n points on the circle and join all pairs, the maximum number of parts got is

$$\frac{1}{24}n(n-1)(n-2)(n-3)+\frac{1}{2}n(n-1)+1$$

The interesting point is that in this expression and in 2^{n-1}, if we take $n=$ 1, 2, 3, 4, 5 we get the same numbers 1,2,4,8,16. From $n = 6$ onwards, the numbers differ.

In other words, all these have the general algebraic form

$$x_n = an + b$$

where a and b are specified numbers.

Are all arithmetic sequences of this form? If we take the first term of an arithmetic sequence as f and the common difference as d, then its terms are

$$f,\ f+d,\ f+2d,\ \dots$$

In general, its n^{th} term would be

$$f+(n-1)d = dn + (f-d)$$

This means multiplying each n by the number d and adding the number $f-d$.

For example, in the arithmetic sequence with first term 2 and common difference 7, the n^{th} term is

$$2+7(n-1) = 7n-5$$

That is, $x_n = 7n - 5$, On the other hand, it is not difficult to see that any sequence $x_n = an + b$ is an arithmetic sequence. Taking $n = 1$, 2, 3, . . . in this, we get the terms of the sequence as

$$a+b,\ 2a+b,\ 3a+b,\ \dots$$

and we can easily see that it is indeed an arithmetic sequence with first term $a+b$ and common difference a.

Every arithmetic sequence is of the form $x_n = an + b$; conversely, every sequence of this form is an arithmetic sequence.

Now some problems based on these ideas:

- The first term and common difference of some arithmetic sequences are given below. Write each of them in the form $x_n = an + b$; also write down the first three terms of each:
    - first term −2, common difference 5
    - first term 2, common difference −5
    - first term 1, common difference $\frac{1}{2}$

- The algebraic form of some sequences are given below. Check whether each of them is an arithmetic sequence or not; also find out the first term and the common difference of the arithmetic sequences.
    - $x_n = 4 - 3n$
    - $x_n = n^2 + 2$
    - $x_n = \frac{n+1}{2}$
    - $x_n = \frac{n+2}{n}$
    - $x_n = (n + 1)^2 - (n - 1)^2$
- Multiply each odd number by 2 and add 1. If these numbers are written in order, does it form an arithmetic sequence? Write the algebraic form of this sequence. Now consider the odd numbers not in this sequence. Do they form an arthimetic sequence? What is its algebraic form?
- Consider the arithmetic sequence with first term $\frac{1}{2}$ and common difference $\frac{1}{4}$. Is 1 a term of this sequence? What about 2?

  Write the algebraic form of this sequence. Prove that all natural numbers occur in this sequence.
- Consider the arithmetic sequence with first term $\frac{1}{2}$ and common difference $\frac{1}{3}$. Is 1 a term of this sequence? What about 2?

  Write the algebraic form of this sequence. Prove that no natural number occurs in this sequence.
- In an arithmetic sequence, the ratio of the first term to the second is 2 : 3. What is the ratio of the third term to the fifth term?

Sums

We have seen a trick to compute the sum of consecutive natural numbers in the section **Triangular numbers,** of the lesson **Polynomials** in the Class 9 textbook. Let's take a closer look at it. For example, suppose we want to find the sum of the natural numbers from 1 to 10. (We can actually add the numbers to get the sum. But what about the sum from 1 to 100? The trick we are about to give can be used to find this also.)

Language of laws

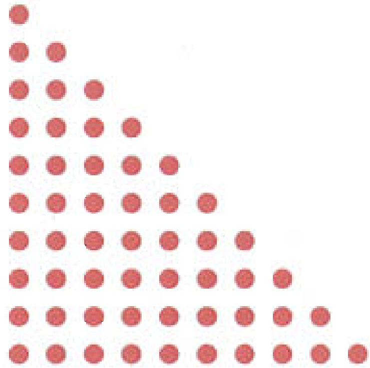
We have noted that to find the terms of a sequence, the law of formation should be specified. And we have seen some examples of how such rules can be written in algebra.

But not all sequences can be algebraically described. For example, no algebraic formula to find the n^{th} prime number has been discovered; that is, no formula to directly compute the number at a specified position in the sequence 2, 3, 5, 7, 11, 13, . . . of primes is known.

Again, there is no algebraic formula to compute the digit at a specified location in the sequence 3, 1, 4, 1, 5, 9, . . . got from the decimal representation of π.

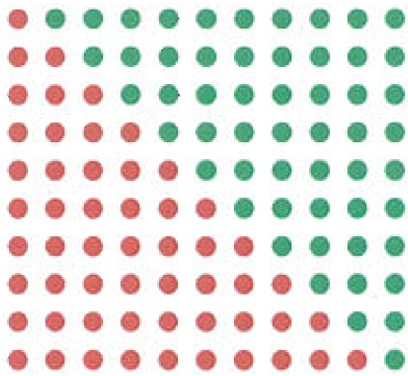
In such cases, we can only specify the rule of forming the terms in ordinary language.

This sum can be visualized as the number of dots in the picture below:



Let's make another copy of this triangle:

and join it upside down with the first



How many dots are there in all, red and green, in this rectangle?

There are 10 rows and each row has 11 dots; so $10 \times 11 = 110$.

This is twice the sum we want. So,

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \frac{1}{2} \times 10 \times 11 = 55$$

Metamorphosis

We saw that the algebraic form of an arithmetic sequence is

$$x_n = an + b$$

What does this mean?

If we multiply all natural numbers by a specific number and then add a specific number, we get an arithmetic sequence. For example, multiplying by $\frac{1}{2}$ and adding -1 gives the arithmetic sequence, $-\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots$ (Its algebraic form is $x_n = \frac{1}{2}n - 1$.)

Also, every arithmetic sequence arises this way. For example, the arithmetic sequence 7, 16, 25, . . . has the algebraic form $x_n = 9n - 2$ and this means multiplying all natural numbers by 9 and adding -2.

This process can be done using numbers instead of pictures: if we write

$$s = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

then

$$\begin{aligned} 2s &= (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10) + \\ &\quad (10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1) \\ &= 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 \\ &= 10 \times 11 \\ &= 110 \end{aligned}$$

and so

$$s = \frac{1}{2} \times 110 = 55$$

We can use the same argument, whether we add up to 10 or 100. In general

The sum of all natural numbers from one to a specific natural number is half the product of the number itself with the next natural number.

In algebraic language,

$$1 + 2 + 3 + \ldots + n = \frac{1}{2} n(n + 1)$$

Using this, we can find the sum of other arithmetic sequences also. See these examples:

- How do we find the sum of the even numbers 2, 4, 6, . . . , 100?

  Even numbers are got by multiplying natural numbers by 2. So,

  $$2 + 4 + 6 + \ldots + 100 = 2(1 + 2 + 3 + \ldots + 50)$$

  As we saw just now,

  $$1 + 2 + 3 + \ldots + 50 = \frac{1}{2} \times 50 \times 51$$

  So, we have

  $$2 + 4 + 6 + \ldots + 100 = 2 \times \frac{1}{2} \times 50 \times 51 = 2550$$

A math tale

We have mentioned the mathematician Gauss in the Class 9 textbook. It is said that he showed extraordinary mathematical ability right from an early age.

This happened when Gauss was just ten and in school. His teacher asked the children to add all numbers from 1 to 100, just to keep them quiet. Young Gauss did it in a flash, explaining his anwer thus: "1 and 100 make 101, so do 2 and 99; there are 50 such pairs and so the answer is $50 \times 101 = 5050$."

A sequence from the past

We have mentioned the ancient mathematical manuscript called the Ahmose Papyrus in the section **Ancient math,** of the lesson **Equations** in the Class 8 textbook. Problem 64 in it goes something like this:

10 hekats of barley is to be distributed among 10 people in a definite order. Each should get $\frac{1}{8}$ hekat more than the previous one. How much should be given to each?

Here *hekat* is a measure used in those days. The answer to this is given in the papyrus as follows:

1. Divide 10 by 10. We get 1
2. Subtract 1 from 10 and multiply it by half of $\frac{1}{8}$. We get $\frac{9}{16}$
3. Add this to the 1 got in the first step. The number $1\frac{9}{16}$ got now is the largest share.
4. Subtract $\frac{1}{8}$ from this repeatedly to get the other shares.

What is the logic behind this computation?

How do we do it using current techniques?

- We want to find the sum of a specified number of odd numbers 1, 3, 5, ···

How do we write this sequence of odd numbers in algebra?

The n^{th} term of this sequence is

$$1 + (n - 1) \times 2 = 2n - 1$$

and we can write it as

$$x_n = 2n - 1$$

So, the sum of the first n odd numbers

$$= x_1 + x_2 + x_3 + \dots + x_n$$

$$= (2 \times 1 - 1) + (2 \times 2 - 1) + (2 \times 3 - 1) \dots + (2n - 1)$$

$$= 2(1 + 2 + 3 + \dots + n) - \overbrace{(1+1+1+\dots+1)}^{n\,times}$$

$$= \left(2 \times \frac{1}{2}n(n + 1)\right) - n$$

$$= n(n + 1) - n$$

$$= n^2$$

Thus the sum of a specified number of consecutive odd numbers starting from 1 is equal to the square of the number of odd numbers added. (This was noted in the lesson **Square Numbers** in the Class 7 textbook.)

We can find the sum of a specified number of consecutive terms of any arithmetic sequence in the same way as we found the sum of odd numbers.

We can write the general term of any arithmetic sequence as

$$x_n = an + b$$

So,

$$x_1 + x_2 + x_3 + \dots + x_n$$

$$= (a \times 1 + b) + (a \times 2 + b) + (a \times 3 + b) + \dots + (an + b)$$

$$= a(1 + 2 + 3 + \dots + n) + \overbrace{(b + b + b + \dots + b)}^{n\,times}$$

$$= \left(a \times \frac{1}{2}n(n + 1)\right) + (n \times b)$$

$$= \frac{1}{2}an(n + 1) + bn$$

For convenience, we rewrite this in a slightly different way:

$$\frac{1}{2}an(n+1) + bn = \frac{1}{2}n\big(a(n+1) + 2b\big)$$

$$= \frac{1}{2}n\big((an+b) + (a+b)\big)$$

$$= \frac{1}{2}n(x_n + x_1)$$

What is the meaning of this?

The sum of a specified number of consecutive terms of an arithmetic sequence is half the product of the number of terms with the sum of the first and the last terms.

For example, suppose we want to find the sum of the first 50 terms of the sequence 3, 5, 7, ... We first find the 50^{th} term as

$$3 + (49 \times 2) = 101$$

By what we have seen above, we can compute the sum as

$$\frac{1}{2} \times 50 \times (3 + 101) = 2600$$

Now try these problems on your own:

- The first term of an arithmetic sequence is 5 and the common difference is 2. Find the sum of its first 25 terms.
- Find an algebraic expression to compute the sum of the first n terms of the arithmetic sequence with first term f and common difference d.
- If $5^2 \times 5^4 \times 5^6 \times ... \times 5^{2n} = (0.04)^{-28}$ what is n?
- Find the sum of all three digit numbers which are multiples of 9.
- Prove that the sum of the first five terms of any arithmetic sequence is five times the middle number. What about the sum of the first seven terms? Can you formulate a general principle from these examples?

Another way

There is another technique for finding the sum of consecutive natural numbers.

We know that for any number x,

$$(x+1)^2 - x^2 = 2x + 1$$

Taking $x = 1, 2, 3, \ldots, n$ in this, we get

$$2^2 - 1^2 = (2 \times 1) + 1$$

$$3^2 - 2^2 = (2 \times 2) + 1$$

$$4^2 - 3^2 = (2 \times 3) + 1$$

....

$$(n+1)^2 - n^2 = (2 \times n) + 1$$

What if we add all these equations? We get

$$(n+1)^2 - 1 = 2(1 + 2 + 3 + ... + n) + n$$

From this, we find

$$1 + 2 + 3 + ... + n$$

$$= \frac{1}{2}((n+1)^2 - 1 - n)$$

$$= \frac{1}{2}(n^2 + n)$$

$$= \frac{1}{2}n(n+1)$$

Sum of squares

We can use an algebraic identity to compute the sum of the squares of natural numbers, just as we found the sum of natural numbers. We have seen the identitiy

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1$$

(See the section **Patterns of power**, of the lesson **Polynomials** in the Class 9 textbook.)

From this we can see that for any number x,

$$(x+1)^3 - x^3 = 3x^2 + 3x + 1$$

As before, if we take $x = 1, 2, 3, ..., n$ in this and add, we get

$$(n+1)^3 - 1 = 3(1^2 + 2^2 + 3^2 + ... + n^2) + 3(1 + 2 + 3 + ... + n) + n$$

That is,

$$n^3 + 3n^2 + 3n$$
$$= 3(1^2 + 2^2 + 3^2 + ... + n^2) + \frac{3}{2}n(n+1) + n$$

So, we get

$$1^2 + 2^2 + 3^2 + ... + n^2$$
$$= \frac{1}{3}\left(n^3 + 3n^2 + 3n - \frac{3}{2}n(n+1) - n\right)$$

Simplifying the right side of this equation, we get

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

- The sum of the first n terms of an arithmetic sequence is $2n^2 + 3n$. Find the algebraic form of this sequence.

Do the problems below in your head:

- Consider the arithmetic sequences 3, 5, 7, . . . and 4, 6, 8, . . . How much more is the sum of the first 25 terms of the second than the sum of the first 25 terms of the first?
- How much more is the sum of the natural numbers from 21 to 40 than the sum of the natural numbers from 1 to 20?
- $51 + 52 + 53 + ... + 70$
- $\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + ... + \frac{25}{2}$
- $\frac{1}{2} + 1 + 1\frac{1}{2} + 2 + 2\frac{1}{2} + ... + 12\frac{1}{2}$

Project

- Prove that if in an arithmetic sequence of natural numbers, one of the terms is a perfect square, then there are many terms which are perfect squares. Is there an arithmetic sequence of natural numbers in which no term is a perfect square?