MATHEMATICAL SCIENCES

Paper – III

SECTION – I

Note : i) Answer both the questions.

ii) Each question carries twenty marks. $2 \times 20 = 40$

1. (a) Discuss the cubic spline interpolation by using Hermite cubic interpolant and apply it to find cos (3·14159) for free boundary conditions by using the following data :

x :	0	1	3	3·5	5
cos x :	1	0·54030	– 0·98999	– 0·93646	0·28366

(b) Deduce the minimizing property of cubic splines.

OR

Show that a set $M \subset C[a, b]$ is compact in $C[a, b]$ if and only if the aggregate of functions $x(t) \in M$ are uniformly bounded and equi-continuous.

OR

Derive the steady-state equation of the multiserver Merkovian model (M/M/C) and obtain its solution.

2. A homogeneous solid sphere of radius R has the initial temperature distribution $f(r)$, $0 \le r \le R$, where r is the distance measured from the centre. The surface temperature is maintained at 0°. Show that the temperature $T(r, t)$ in the sphere is the solution of

$$T_t = c^2\left(T_{rr} + \frac{2}{r} T_r\right)$$

where c^2 is a constant. Show that the temperature in the sphere for $t > 0$ is given by

$$T(r, t) = \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R} r\right) exp\left(-\lambda_n^2 t\right), \lambda_n = \frac{c\, n\pi}{R}.$$

OR

A rigid body is set rotating under no forces (moment of finite forces about the principal axes being zero) about its one point with angular velocity components $\omega_j = n$, $\omega_2 = 0$, $\omega_3 = n\sqrt{2}$ about the principal axes, respectively. If the respective principal moments are $4A$, $3A$ and A, respectively then discuss the ultimate motion.

OR

What is sampling distribution ? Derive non-central t-distribution.

SECTION – II

Note : i) Answer all questions.

ii) Each question carries fifteen marks. $3 \times 15 = 45$

3. (a) Suppose the function $f(z)$ is analytic everywhere in a closed domain *D*, except at a finite number of isolated singularities z_k ($k = 1, 2, \ldots, n$) lying inside the domain *D*. Then show that

$$\int_{\Gamma^+} f(\rho)\, d\rho = 2\pi i \sum_{k=1}^{n} Res\,[f(z), z_k]$$

where Γ^+ is the complete boundary of domain *D* traversed in the positive direction and hence evaluate the integral

$$I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}, \quad |a| < 1.$$

(b) Construct a function that maps the strip $0 < Re\, z < a$ conformally onto the upper half-plane $Im\, \omega > 0$.

OR

Find the Hamilton's canonical equations of motion of a particle of mass *m* moving in a force field of potential $V(\rho, \phi, z)$ in cylindrical polar co-ordinates (ρ, ϕ, z).

OR

Show that every Bernoulli sequence of *r.v.s.* obeys the weak law of large numbers.

4. Prove that the family *M* of Lebesgue measurable sets is an algebra.

OR

Give two examples of non-parametric tests. Discuss the exact and the limiting null distributions of the corresponding test statistics.

OR

Find the rate of convergence of Newton-Raphson method to find the root of an equation $f(x) = 0$.

5. Show that the integral equation

$$y(x) = \int_0^x (x+t)\, y(t)\, dt + 1$$

is equivalent to the differential equation

$$y''(x) - 2xy'(x) - 3y(x) = 0$$

$$y(0) = 1, \quad y'(0) = 0.$$

OR

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from Poisson distribution with parameter θ. The natural conjugate prior for θ is Gamma (α, β). Then obain the posterior density of θ.

OR

Using Ritz method based on the variational principle, show that the approximate solution of the boundary value problem

$$y'' + y = x, \quad y(0) = y(1) = 0$$

is $y = \frac{5}{18}\left(-x + x^2\right)$.

SECTION – III

Note : i) Answer all questions.

ii) Each question carries ten marks. $9 \times 10 = 90$

6. Show that a finite integral domain is a field.

7. In a plane triangle, find the maximum value of cos A cos B cos C.

8. Find the shortest distance between the parabola $y = x^2$ and the straight line $x - y = 5$, using calculus of variation.

9. Define conformal mapping. What are essential conditions for conformal transformation ? Examine that following transformations are everywhere conformal or not and determine critical points :

(i) $f(z) = (z - 1)^2$

(ii) $f(z) = \dfrac{z - i}{z + i}$.

10. Use Cayley-Hamilton theorem to find A^{-1}, where

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}.$$

11. Find the eigenvalue and eigenfunctions of the following homogeneous integral equation with degenerate kernels

$$y(x) = \lambda \int_0^1 \left(2xt - 4x^2\right) y(t)\, dt.$$

12. Explain the principle of likelihood ratio test.

13. Define a BIBD and state the situations in which such designs are used.

14. Give the circumstances under which systematic sampling is to be preferred to simple random sampling.

SECTION – IV

Note : i) Answer all questions.

ii) Each question carries five marks. $5 \times 5 = 25$

15. If the vectors $(0, 1, a)$, $(1, a, 1)$, $(a, 1, 0)$ of the vector space $R^3(R)$ be linearly dependent, then find the value of a.

16. If the function $f(z) = \frac{iz}{2}$ is defined on the open disk $|z| < 1$, show that $\lim_{z \to 1} f(z) = \frac{i}{2}$, the point $z = 1$ being on the boundary of definition.

17. Find the general solution of

$$\left(x^2 + 1\right) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = 6\left(x^2 + 1\right)^2,$$

given that $y = x$ and $y = x^2 - 1$ are linearly independent solutions of corresponding homogeneous equation.

18. Find the curve for which the surface of revolution is minimum.

19. There are two identical urns containing respectively 4 white, 3 red balls and 3 white, 7 red balls. An urn is chosen at random and a ball is drawn from it. Find the probability that the ball is white. If the ball drawn is white, what is the probability that it is from the first urn ?